

LEIBNIZ SEMINORMS FOR “MATRIX ALGEBRAS CONVERGE TO THE SPHERE”

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ABSTRACT. In an earlier paper of mine relating vector bundles and Gromov–Hausdorff distance for ordinary compact metric spaces, it was crucial that the Lipschitz seminorms from the metrics satisfy a strong Leibniz property. In the present paper, for the now non-commutative situation of matrix algebras converging to the sphere (or to other spaces) for quantum Gromov–Hausdorff distance, we show how to construct suitable seminorms that also satisfy the strong Leibniz property. This is in preparation for making precise certain statements in the literature of high-energy physics concerning “vector bundles” over matrix algebras that “correspond” to monopole bundles over the sphere. We show that a fairly general source of seminorms that satisfy the strong Leibniz property consists of derivations into normed bimodules. For matrix algebras our main technical tools are coherent states and Berezin symbols.

INTRODUCTION

In a previous paper [29] I showed how to give a precise meaning to statements in the literature of high-energy physics and string theory of the kind “Matrix algebras converge to the sphere”. (See [29] for numerous references to the relevant physics literature.) I did this by introducing the concept of “compact quantum metric spaces”, in which the metric data is given by a seminorm on the non-commutative “algebra of functions”. This seminorm plays the role of the usual Lipschitz seminorm on the algebra of continuous functions on an ordinary compact metric space. However, I was somewhat puzzled by the fact that I needed virtually no algebraic conditions on the seminorm, only an important analytic condition. But when I later began trying to give precise meaning to further statements in the physics literature of the

1991 *Mathematics Subject Classification.* Primary 46L87; Secondary 53C23, 58B34, 81R15, 81R30.

Key words and phrases. quantum metric space, Gromov–Hausdorff distance, Lipschitz, Leibniz seminorm, coadjoint orbits, coherent states, Berezin symbols.

The research reported here was supported in part by National Science Foundation grant DMS-0500501.

kind “here are the vector bundles over the matrix algebras that correspond to the monopole bundles over the sphere” (see [31] for many references), I found that for ordinary metric spaces a strong form of the Leibniz inequality for the seminorm played a crucial role [31]. (See, for example, the proof of proposition 2.3 of [31].) However, on returning to the non-commutative case of matrix algebras converging to the sphere (or to other spaces), for some time I did not see how to construct useful seminorms that brought the matrix algebras and sphere close together while also having the strong Leibniz property. The main purpose of this paper is to show how to construct such seminorms. As in the earlier paper [29], the setting is that of coadjoint orbits of compact semisimple Lie groups, of which the 2-sphere is the simplest example. The main technical tools continue to be coherent states and Berezin symbols.

In the first four sections of this paper we show that a fairly general setting for obtaining seminorms that possess the strong Leibniz property that we need consists of derivations into normed bimodules, and we examine various aspects of this topic. The strong Leibniz property for a seminorm L on a normed unital algebra A consists of the usual Leibniz inequality together with the inequality

$$L(a^{-1}) \leq \|a^{-1}\|^2 L(a)$$

whenever a is invertible in A . I have not seen this latter inequality discussed in the literature. In Section 4 we put together the various conditions that we have found to be important, and there-by give a tentative definition for a “compact C^* -metric space”.

In Section 5 we examine the use of seminorms with the strong Leibniz property in connection with quantum Gromov–Hausdorff distance. (I expect that many of the ideas and techniques developed in this paper will apply to many other classes of examples beyond “Matrix algebras converge to the sphere”.) In Section 6 we extend to the case of strongly Leibniz seminorms the construction technique introduced in [28] that we called “bridges”. Sections 7 and 8 contain those pieces of our development that can be carried out for certain homogeneous spaces of any compact group (including finite ones). Section 9 gives the statement of our main theorem for coadjoint orbits, while Sections 10 through 13 contain the detailed technical development needed to prove our main theorem. Finally, in Section 14 we relate our results to other variants of quantum Gromov–Hausdorff distance that have been developed by David Kerr, Hanfeng Li, and Wei Wu [13, 14, 17, 18, 38, 39, 40].

We can describe our basic setup and our main theorem somewhat more specifically as follows, where definitions for various terms are given in later sections. Let G be a compact semisimple Lie group,

let (U, \mathcal{H}) be an irreducible unitary representation of G , and let P be the rank-one projection along a highest weight vector for (U, \mathcal{H}) . Let α be the action of G on $\mathcal{L}(\mathcal{H})$ by conjugation by U , and let H be the α -stability group of P . Let $A = C(G/H)$. Let ω be the highest weight for U , and for each $n \in \mathbb{Z}_{>0}$ let (U^n, \mathcal{H}^n) be the irreducible representation of G of highest weight $n\omega$. Let α also denote the action of G on $B^n = \mathcal{L}(\mathcal{H}^n)$ by conjugation by U^n .

Choose on G a continuous length-function ℓ . Then ℓ and the translation action of G on A , as well as the actions α of G on each B^n , determine seminorms L_A on A and L_{B^n} on B^n that make (A, L_A) and each (B^n, L_{B^n}) into compact C^* -metric spaces.

Main Theorem (sketchy statement of Theorem 9.1). *For any $\varepsilon > 0$ there exists an N such that for any $n \geq N$ we can explicitly construct a strongly Leibniz seminorm, L_n , on $A \oplus B^n$ making $A \oplus B^n$ into a compact C^* -metric space, such that the quotients of L_n on A and B^n are L_A and L_{B^n} , and for which the quantum Gromov-Hausdorff distance between A and B^n is no greater than ε .*

I plan to apply the results of this paper in a future paper to discuss vector bundles over non-commutative spaces (e.g., monopole bundles), along the lines used for ordinary spaces in [31].

I developed part of the material presented here during a ten-week visit at the Isaac Newton Institute in Cambridge, England, in the Fall of 2006. I am very appreciative of the stimulating and enjoyable conditions provided by the Isaac Newton Institute.

I am grateful to Hanfeng Li for some important comments on the first version of this paper, which led to some substantial improvements given in the present version.

1. STRONGLY LEIBNIZ SEMINORMS

From my investigation of the relation between vector bundles on compact metric spaces that are close together, both for ordinary spaces [31] and for non-commutative spaces (a continuing investigation), I have found that the following properties are very important when considering the seminorms that play the role of the Lipschitz seminorms of ordinary metric spaces. Unless the contrary is stated, we allow our seminorms to take the value $+\infty$, but we require that they take value 0 at 0. We use the usual conventions for calculating with $+\infty$. The following definition is close to definition 2.1 of [31].

Definition 1.1. Let A be a normed unital algebra over \mathbb{R} or \mathbb{C} , and let L be a seminorm on A . We say that:

- 1) L is *Leibniz* if it satisfies the inequality

$$L(ab) \leq L(a)\|b\| + \|a\|L(b)$$

for all $a, b \in A$.

- 2) L is *strongly Leibniz* if it is Leibniz, and $L(1) = 0$, and if for any $a \in A$ that has an inverse in A , we have

$$L(a^{-1}) \leq \|a^{-1}\|^2 L(a).$$

- 3) L is *finite* if $L(a) < \infty$ for all $a \in A$.
 4) L is *semifinite* if $\{a : L(a) < \infty\}$ is norm-dense in A .
 5) L is *continuous* if it is norm-continuous.
 6) L is *lower semicontinuous* if for one $r \in \mathbb{R}_{>0}$, hence for all $r > 0$, the set

$$\{a \in A : L(a) \leq r\}$$

is norm-closed in A .

If, furthermore, A is a $*$ -normed algebra (i.e., has an isometric involution), then we define L^* by $L^*(a) = L(a^*)$ for $a \in A$. We then say that L is a $*$ -seminorm if $L = L^*$.

The proof of the following proposition is straightforward.

Proposition 1.2. *Let A be a unital normed algebra.*

- i. *Let L be a seminorm on A and let $r \in \mathbb{R}^+$. If L satisfies one of the properties 1–6 above then rL satisfies that same property.*
- ii. *Let L_1 and L_2 be two seminorms on A . If they are both Leibniz, or strongly Leibniz, or finite, or continuous, or lower semicontinuous, then so is $L_1 + L_2$.*
- iii. *Let $\{L_\alpha\}$ be a family of seminorms on A , possibly infinite, and let L be the supremum of this family. (I.e., $L = \bigvee_\alpha L_\alpha$, defined by $L(a) = \sup_\alpha \{L_\alpha(a)\}$. For two seminorms, L and L' , we will denote their maximum by $L \vee L'$.) Then L is a seminorm on A , and if each L_α is Leibniz, or strongly Leibniz, or lower semicontinuous, then so is L .*
- iv. *If A is a $*$ -normed algebra and if L satisfies one of the properties 1–6 above, then L^* satisfies that same property.*

I have seen no discussion of the strong Leibniz property in the literature. I do not know of an example of a finite Leibniz seminorm which does not satisfy the inequality for $L(a^{-1})$ in the definition of “strongly Leibniz”. But if we allow the value $+\infty$ then examples can be constructed in the following way. Let A be a unital normed algebra and let B be a unital subalgebra of A . Let L_0 be a finite Leibniz seminorm on B . Define a Leibniz seminorm, L , on A by $L(a) = L_0(a)$ if $a \in B$

and $L(a) = +\infty$ otherwise. If B contains an element that is invertible in A but not in B then L is not strongly Leibniz. For example, let $A = C([0, 1])$ and let B be its subalgebra of polynomial functions, with $L_0(f) = \|f'\|$. (This example is not lower semicontinuous.)

Let $A^f = \{a : L(a) < \infty\}$. It is clear that if L is Leibniz then A^f is a subalgebra of A . If L is in fact strongly Leibniz and $a \in A^f$, then clearly a is invertible in A^f if and only if it is invertible in A . It follows that for any $a \in A^f$ the spectrum of a in A^f will be the same as its spectrum in A . In stupid examples we may have $1 \notin A^f$, but with that understood, we see that:

Proposition 1.3. *If L is strongly Leibniz then A^f is a spectrally stable subalgebra of A .*

The importance of this proposition will be seen in Section 3. We also remark that if A has an involution and if L is a Leibniz seminorm that is also a $*$ -seminorm, then A^f is a $*$ -subalgebra of A .

Simple arguments prove the following two propositions.

Proposition 1.4. *Let A be a normed unital algebra, and let L be a seminorm on A . Let B be a unital subalgebra of A , equipped with the norm from A . If L is Leibniz, or strongly Leibniz, or finite, or continuous, or lower semicontinuous, then so is the restriction of L to B as a seminorm on B . (But if L is semifinite, its restriction to B need not be semifinite.)*

Proposition 1.5. *Let A be a $*$ -normed unital algebra and let L be a seminorm on A . Let $\tilde{L} = L \vee (L^*)$. Then \tilde{L} is a $*$ -seminorm. If L is Leibniz, or strongly Leibniz, or finite, or continuous, or lower semicontinuous, then so is \tilde{L} . (But if L is semifinite, \tilde{L} need not be semifinite.)*

So in this way we can usually arrange to work with $*$ -seminorms when dealing with $*$ -algebras.

Here is another way to combine seminorms:

Proposition 1.6. *Let L_1, \dots, L_n be seminorms on a normed unital algebra A , and let $\|\cdot\|_0$ be a norm on \mathbb{R}^n with the property that if $(r_j), (s_j) \in \mathbb{R}^n$, and if $0 \leq r_j \leq s_j$ for all j , $1 \leq j \leq n$, then $\|(r_j)\|_0 \leq \|(s_j)\|_0$. Define a seminorm, N , on A by $N(a) = \|(L_j(a))\|_0$, with the evident meaning if $L_j(a) = \infty$ for some j . If each L_j satisfies a particular one of properties 1, 2, 3, 5, 6 of Definition 1.1 then N satisfies that property too.*

Proof. If each L_j is Leibniz, then

$$N(ab) = \|(L_j(ab))\|_0 \leq \|(L_j(a)\|b\| + \|a\|L_j(b))\|_0$$

$$\leq \|(L_j(a))\|_0 \|b\| + \|a\| \|(L_j(b))\|_0 = N(a) \|b\| + \|a\| N(b),$$

and if each L_j is strongly Leibniz, then also

$$N(a^{-1}) = \|(L_j(a^{-1}))\|_0 \leq \|(\|a^{-1}\|^2 L_j(a))\|_0 = \|a^{-1}\|^2 N(a).$$

It is clear that if each L_j is finite, or continuous, then so is N .

Suppose instead that each L_j is lower semicontinuous. Let (a_m) be a sequence in A which converges in norm to $a \in A$, and suppose that there is a constant, K , such that $N(a_m) \leq K$ for each m . For each m let $p^m = (L_j(a_m)) \in \mathbb{R}^n$, so that $\|p^m\|_0 \leq K$. Since the K -ball of \mathbb{R}^n for $\|\cdot\|_0$ is compact, we can pass to a convergent subsequence, so we can assume that the sequence $\{p^m\}$ converges to a vector, p , in \mathbb{R}^n such that $\|p\|_0 \leq K$ and whose entries are non-negative. Let $\varepsilon > 0$ be given. Then there is an integer m_ε such that if $m \geq m_\varepsilon$ then $L_j(a_m) \leq p_j + \varepsilon$ for each j . Since each L_j is lower semicontinuous, it follows that $L_j(a) \leq p_j + \varepsilon$ for each j . Then $N(a) = \|(L_j(a))\|_0 \leq \|(p_j + \varepsilon)\|_0 \leq \|p\|_0 + \varepsilon \|(1, \dots, 1)\|_0$. Thus $N(a) \leq K$ since $\|p\|_0 \leq K$ and ε is arbitrary. \square

2. GENERAL SOURCES OF STRONGLY LEIBNIZ SEMINORMS

We will now examine general methods for constructing strongly Leibniz seminorms. We recall first [11] that a *first-order differential calculus* over a unital algebra A is a pair (Ω, d) consisting of a bimodule Ω over A and a derivation d from A into Ω , that is, a linear map from A into Ω such that

$$d(ab) = (da)b + a(db)$$

for all $a, b \in \Omega$. (We will always assume that our bimodules are such that 1_A acts as the identity operator on both left and right.) It is common to assume that Ω is generated as a bimodule by the range of d , but we will not need to impose this requirement, though it can always be arranged by replacing Ω by its sub-bimodule generated by the range of d .

Suppose now that A is a normed unital algebra (with $\|1_A\| = 1$), and that (Ω, d) is a first-order differential calculus for A . Assume further that Ω is equipped with a norm that makes it into a normed A -bimodule, that is,

$$\|a\omega b\|_\Omega \leq \|a\| \|\omega\|_\Omega \|b\|$$

for all $a, b \in A$ and $\omega \in \Omega$. We will then say that $(\Omega, d, \|\cdot\|_\Omega)$ is a *normed first-order differential calculus*. We do not require that d be continuous for the norms on A and Ω . We define L on A by

$$L(a) = \|da\|_\Omega.$$

Notice that L is finite, and that L is continuous if d is.

Proposition 2.1. *Let L on A be defined as above for a normed first-order differential calculus. Then L is strongly Leibniz.*

Proof. That L is Leibniz follows immediately from the definitions of a derivation and of a normed bimodule. To see that L is strongly Leibniz, notice first that from the definition of a derivation we obtain $d(1_A) = 0$, so that $L(1_A) = 0$. Suppose now that a is an invertible element of A . Then

$$0 = d(1_A) = d(aa^{-1}) = (da)a^{-1} + a(d(a^{-1})).$$

Thus

$$d(a^{-1}) = -a^{-1}(da)a^{-1}.$$

On taking the norm we see that $L(a^{-1}) \leq \|a^{-1}\|^2 L(a)$. \square

We remark that no effective characterization seems to be known as to which Leibniz seminorms come from normed first-order differential calculi (or of inner ones) as above. They fall within the scope of the “flat” differential seminorms defined in definition 4.3 of [4], and for which equivalent conditions are given in theorem 4.4 of [4]. Necessary conditions for a differential seminorm to be flat are given immediately after definition 4.3 and in proposition 4.7 of [4]. For Leibniz seminorms we see above that a further necessary condition is that of being strongly Leibniz.

Let us now give some examples.

Example 2.2. Let (X, ρ) be a compact metric space. For given $x_0, x_1 \in X$ with $x_0 \neq x_1$ let Ω_{x_0, x_1} be \mathbb{R} or \mathbb{C} according to whether $A = C(X)$ is over \mathbb{R} or \mathbb{C} , and define actions of A on Ω_{x_0, x_1} by

$$f \cdot \omega = f(x_0)\omega, \quad w \cdot f = \omega f(x_1).$$

Define d by

$$df = (f(x_1) - f(x_0))/\rho(x_1, x_0).$$

It is easily checked that (Ω_{x_0, x_1}, d) is a first-order differential calculus over A . Give $A = C(X)$ its supremum norm, $\|\cdot\|_\infty$, and give Ω_{x_0, x_1} the usual norm on \mathbb{R} or \mathbb{C} . Then Ω_{x_0, x_1} is a normed A -bimodule. Clearly d is continuous. We set

$$L_{x_0, x_1}(f) = \|df\| = |f(x_1) - f(x_0)|/\rho(x_1, x_0).$$

Then from Proposition 2.1 it follows that L_{x_0, x_1} is strongly Leibniz (and continuous).

Now let L be the supremum of the L_{x_0, x_1} over all pairs (x_0, x_1) with $x_1 \neq x_0$. We obtain in this way the usual Lipschitz seminorm, L^ρ , on $C(X)$. From Proposition 1.2 it follows that L^ρ is strongly Leibniz and

lower semicontinuous. Of course L^ρ is not continuous in general. But L^ρ is semifinite, since it is finite on the functions $f_{x_0}(x) = \rho(x, x_0)$, and these already generate a dense subalgebra, as seen by means of the Stone–Weierstrass theorem.

This example can be recast in a quite familiar form as follows. Let $Z = (X \times X) \setminus \Delta$ where Δ is the diagonal of $X \times X$. Thus Z is a locally compact space. Let $\Omega = C_b(Z)$, the linear space of bounded continuous functions on Z with the supremum norm. Then Ω is a normed $C(X)$ -bimodule for the actions

$$(f\omega)(x_0, x_1) = f(x_0)\omega(x_0, x_1), \quad (\omega f)(x_0, x_1) = \omega(x_0, x_1)f(x_1).$$

Let A denote the subalgebra of $C(X)$ consisting of the Lipschitz functions, and define a derivation d from A to $C(Z)$ by

$$(df)(x_0, x_1) = (f(x_1) - f(x_0))/\rho(x_1, x_0).$$

Then the usual Lipschitz seminorm is given by $L^\rho(f) = \|df\|_\infty$. Alternatively, let $\Omega = C(Z)$, the space of all continuous, possibly unbounded, functions on Z , as a $C(X)$ -bimodule in the above way. Then d can be defined on all of $C(X)$ by the above formula. We can now consider the supremum norm on $C(Z)$, taking value $+\infty$ on unbounded functions (so a bit beyond our definitions above), and again set $L^\rho(f) = \|df\|_\infty$.

Example 2.3. Let us now consider some examples in which the normed unital algebra A may be non-commutative. If Ω is an A -bimodule, one always has the corresponding inner derivations. That is, if $\omega \in \Omega$ we can set $d^\omega(a) = \omega a - a\omega$. If Ω is a normed A -bimodule then d^ω is continuous, with $\|d^\omega\| \leq 2\|\omega\|$. The corresponding seminorm, L^ω , defined by $L^\omega(a) = \|d^\omega(a)\|$, is then a continuous strongly Leibniz seminorm.

Suppose now that B is a unital normed algebra and that π is a unital homomorphism from A into B . Then we can view B as a bimodule over A in the evident way, and obtain inner derivations and corresponding strongly Leibniz seminorms, which are continuous if π is.

Example 2.4. Suppose now that π is a non-degenerate representation of A as operators on a normed space X , so that π can be viewed as a unital homomorphism from A into $B(X)$, the algebra of bounded operators on X . Then $B(X)$ can be viewed in the evident way as a bimodule over A , and any element, D , of $B(X)$ determines an inner derivation, and corresponding seminorm

$$L(a) = \|D\pi(a) - \pi(a)D\| = \|[D, \pi(a)]\|.$$

More generally, if one has two representations, π^1 and π^2 of A on X , then one can view $B(X)$ as an A -bimodule via

$$a \cdot T \cdot b = \pi^1(a)T\pi^2(a),$$

and again any element, D , of $B(X)$ will determine an inner derivation. (The twisted commutators in equation 2.4 and lemma 2.2 of [8] fit into this view, except that there D is usually an unbounded operator.) Alternately one can assemble π^1 and π^2 into one representation on $X \oplus X$, and use the operator $\begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix}$ on $X \oplus X$.

As an important particular case, for X we can take A itself and let π be the left-regular representation of A on itself. As element of $B(X)$ we can take an isometric algebra automorphism, α , of A . Then

$$\begin{aligned} (\alpha \circ \pi(a) - \pi(a) \circ \alpha)(b) &= \alpha(ab) - a\alpha(b) \\ &= (\alpha(a) - a)\alpha(b). \end{aligned}$$

From this we see that

$$\|\alpha \circ \pi(a) - \pi(a) \circ \alpha\| = \|\alpha(a) - a\|,$$

so that if we set $L(a) = \|\alpha(a) - a\|$ then L will be a continuous strongly Leibniz seminorm. We can view this in another way. View A as a bimodule over A by

$$a \cdot b \cdot c = ab\alpha(c),$$

and set $d(a) = \alpha(a) - a$. It is easily checked that d is a (continuous) derivation, and so from Proposition 2.1 we see again that L is strongly Leibniz. (This does not require that α be isometric.)

Example 2.5. Now let G be a group, and let α be an action of G on A , that is, a homomorphism from G into $\text{Aut}(A)$. Let ℓ be a length-function on G . For each $x \in G$ with $x \neq e_G$ the map $a \mapsto \|\alpha_x(a) - a\|/\ell(x)$ is a continuous strongly Leibniz seminorm. Let L be the supremum over G of all of these seminorms, so that

$$L(a) = \sup\{\|\alpha_x(a) - a\|/\ell(x) : x \neq e_G\}.$$

By Proposition 1.2 we see that L is a lower-semicontinuous strongly-Leibniz seminorm. Of course L may not be semifinite. But if G is a locally compact group, if A is complete, so a Banach algebra, if α is a strongly continuous action by isometric automorphisms of A , and if ℓ is a continuous length-function, then the discussion before theorem 2.2 of [27] shows that L is semifinite. The discussion there is stated just for C^* -algebras, but it applies without change to Banach algebras.

Example 2.6. Suppose now that G is a connected Lie group, and that α is a strongly continuous action of G on A by isometric automorphisms. Let \mathfrak{g} denote the Lie algebra of G , and let A^∞ denote the dense subalgebra of smooth elements of A for the action α . We let α also denote the corresponding infinitesimal action of \mathfrak{g} on A^∞ , defined by

$$\alpha_X(a) = \left. \frac{d}{dt} \right|_{t=0} \alpha_{\exp(tX)}(a)$$

for $X \in \mathfrak{g}$ and $a \in A^\infty$. The argument in the proof of lemma 3.1 of [27] works here, and shows that

$$\|\alpha_X(a)\| = \sup\{\|\alpha_{\exp(tX)}(a) - a\|/|t| : t \neq 0\}.$$

It follows from Proposition 1.2 that the map $a \mapsto \|\alpha_X(a)\|$ is a finite lower-semicontinuous strongly-Leibniz seminorm on A^∞ . Suppose further that we are given a norm on \mathfrak{g} , and that we set

$$L(a) = \sup\{\|\alpha_X(a)\| : \|X\| \leq 1\}.$$

It follows again from Proposition 1.2 that L is a lower-semicontinuous strongly-Leibniz seminorm on A^∞ , which is easily seen to be finite, but which may well not be norm-continuous.

Example 2.7. Suppose now that G is a Lie group and that (U, \mathcal{H}) is a strongly continuous representation of G on a Hilbert space \mathcal{H} . As discussed in section 3 of [27] we can define an action, α , of G on $\mathcal{B}(\mathcal{H})$ by $\alpha_x(T) = U_x T U_x^*$, and we can let B be the largest subalgebra of $\mathcal{B}(\mathcal{H})$ on which this action is strongly continuous. We can then apply the discussion of the previous example to obtain a seminorm L on B^∞ . If A is a unital $*$ -subalgebra of B^∞ (which need not be carried into itself by α), then according to Proposition 1.4 the restriction of L to A is a lower-semicontinuous strongly-Leibniz $*$ -seminorm which is clearly finite.

Example 2.8. Let us consider the above situation for the special case in which $\mathfrak{g} = \mathbb{R}$. Then U is generated by a self-adjoint (often unbounded) operator, D , on \mathcal{H} , that is, $U_t = e^{itD}$ for all $t \in \mathbb{R}$. Then it follows easily that for $T \in B^\infty$

$$L(T) = \|[D, T]\|,$$

and in particular that the commutator $[D, T]$ is a bounded operator. All of this will then be true for any $T \in A \subset B^\infty$. This applies in particular to the “Dirac” operators on which Connes [6, 11] bases his approach to metric non-commutative differential geometry.

3. CLOSED SEMINORMS

We adapt here some definitions from section 4 of [25]. Let A be a normed unital algebra, and let \bar{A} denote its completion. Let L be a seminorm on A (value $+\infty$ allowed) and let

$$\mathcal{L}_1 = \{a \in A : L(a) \leq 1\}.$$

Let $\bar{\mathcal{L}}_1$ be the closure of \mathcal{L}_1 in \bar{A} , and let \bar{L} denote the corresponding “Minkowski functional” on \bar{A} , defined by setting, for $c \in \bar{A}$,

$$\bar{L}(c) = \inf\{r \in \mathbb{R}^+ : c \in r\bar{\mathcal{L}}_1\}.$$

The value $+\infty$ must be allowed. Then \bar{L} is a seminorm on \bar{A} , and the proof of proposition 4.4 of [25] tells us that if L is lower semicontinuous, then \bar{L} is an extension of L . We call \bar{L} the *closure* of L . We see that the set $\{c \in \bar{A} : \bar{L}(c) \leq 1\}$ is closed in \bar{A} . We say that the original seminorm L on A is *closed* if \mathcal{L}_1 is closed in \bar{A} , or, equivalently, is complete for the norm on A . Clearly if L is closed, then it is lower semicontinuous. If L is closed and is not defined on all of \bar{A} , then \bar{L} is obtained simply by giving it value $+\infty$ on all the elements of \bar{A} that are not in A . It is clear that if L is semifinite then so is \bar{L} . We recall that a unital subalgebra B of a unital algebra A is said to be spectrally stable in A if for any $b \in B$ the spectrum of b as an element of B is the same as its spectrum as an element of A , or equivalently, that any b that is invertible in A is invertible in B .

Proposition 3.1. *Let L be a Leibniz seminorm on a normed unital algebra A . Then \bar{L} is Leibniz. Set*

$$\bar{A}^f = \{c \in \bar{A} : \bar{L}(c) < \infty\}.$$

If $L(1) < \infty$, then \bar{A}^f is a unital spectrally-stable subalgebra of the norm closure of \bar{A}^f in \bar{A} . If A is defined over \mathbb{C} , then \bar{A}^f is stable under the holomorphic-function calculus of its closure.

Proof. Let $c, d \in \bar{A}$. If $\bar{L}(c) = \infty$ or $\bar{L}(d) = \infty$ there is nothing to show for the Leibniz condition. Otherwise, we can find sequences $\{a_n\}$ and $\{b_n\}$ in A such that $\{a_n\}$ converge to c while $\{L(a_n)\}$ converges to $\bar{L}(c)$ and $L(a_n) \leq \bar{L}(c)$ for all n , and similarly for $\{b_n\}$ and d . Then $a_n b_n$ converges to cd and

$$L(a_n b_n) \leq L(a_n) \|b_n\| + \|a_n\| L(b_n) \leq \bar{L}(c) \|b_n\| + \|a_n\| \bar{L}(d),$$

and the right-hand side converges to $\bar{L}(c) \|d\| + \|c\| \bar{L}(d)$. Thus \bar{L} is Leibniz.

If $L(1) < \infty$ so that \bar{A}^f is a unital subalgebra of \bar{A} , it follows from proposition 3.12 of [4] (or proposition 1.7 and theorem 1.17 of [34] or

lemma 1.6.1 of [9]) that \bar{A}^f is spectrally stable in its closure in \bar{A} , and in fact is stable under the holomorphic-function calculus there. We sketch the proof in our simpler setting. Define a new norm, M , on \bar{A}^f by

$$M(c) = \|c\| + \bar{L}(c).$$

Then, as mentioned after definition 4.5 of [25], \bar{A}^f will be complete for the norm M because \bar{L} is closed. (See the proof of proposition 1.6.2 of [37].) Because \bar{L} is Leibniz, M is easily seen to be an algebra norm, so that \bar{A}^f becomes a Banach algebra. Let $c \in \bar{A}^f$. From the Leibniz rule we find that $\bar{L}(c^n) \leq n\|c\|^{n-1}\bar{L}(c)$, so that

$$M(c^n) \leq \|c\|^n + n\|c\|^{n-1}\bar{L}(c).$$

From this it follows that if $\|c\| < 1$ then the series $\sum_{n=0}^{\infty} c^n$ converges for M to an element of \bar{A}^f . Thus $1 - c$ is invertible in \bar{A}^f . It follows that if instead $\|1 - c\| < 1$ then c is invertible in \bar{A}^f . From this it is then easily seen (e.g. lemma 3.38 of [11]) that if $c \in \bar{A}^f$ and if c is invertible in the norm-closure of \bar{A}^f in \bar{A} , then c is invertible in \bar{A}^f . Consequently \bar{A}^f is spectrally stable in its closure in \bar{A} .

Assume now that A is defined over \mathbb{C} . For the definition and properties of the holomorphic-function (or “symbolic”) calculus see [12, 32]. It is well-known that a dense subalgebra that is spectrally stable and is a Banach algebra for a norm stronger than the norm of the bigger algebra, is stable under the holomorphic-function calculus. (See the comments after definition 3.25 of [11].) We briefly recall the reason, for our context. For notational simplicity we assume that \bar{A}^f is dense in \bar{A} . Let $c \in \bar{A}^f$, and let f be a \mathbb{C} -valued function defined and holomorphic on an open neighborhood \mathcal{O} of the spectrum $\sigma_{\bar{A}}(c)$. Let γ be a union of a finite number of curves in \mathcal{O} that surrounds $\sigma_{\bar{A}}(c)$ in the usual way such that the Cauchy integral formula using γ gives f on a neighborhood of $\sigma_{\bar{A}}(c)$. Since \bar{A}^f is spectrally stable in \bar{A} , the function $z \mapsto (z - c)^{-1}$, well-defined on γ , has values in \bar{A}^f . Since \bar{A}^f is a Banach algebra for M , this function is continuous for M , and the integral

$$f(c) = \frac{1}{2\pi i} \int_{\gamma} f(z)(z - c)^{-1} dz$$

is well defined in \bar{A}^f . Since the homomorphism from \bar{A}^f with norm M to \bar{A} with its original norm is clearly continuous, the image of $f(c)$ in \bar{A} will be expressed by the same integral but now interpreted in \bar{A} . But $f(c) \in \bar{A}^f$. So the above integral, but interpreted in \bar{A} , gives an element of \bar{A}^f as was to be shown. \square

For the use of the holomorphic-function calculus when dealing with algebras over \mathbb{R} see proposition 2.4 of [31].

One reason that the property of being closed under the holomorphic-function calculus is important is that it implies that \bar{A}^f and its closure, say B , in \bar{A} have essentially the same finitely-generated projective modules (“vector bundles”) in the sense that any such right module V for B is of the form $V = W \otimes_{\bar{A}^f} B$ for such a right module W over \bar{A}^f , unique up to isomorphism. This is crucial to [31], and to our proposed discussion of projective modules and quantum Gromov–Hausdorff distance for non-commutative C^* -algebras. The inclusion of \bar{A}^f into B also gives an isomorphism of their K -groups. (See appendix IIIC of [6] and theorem 3.44 of [11].)

Proposition 3.2. *Let L be a strongly-Leibniz seminorm on a normed algebra A . Assume that A^f is dense and spectrally stable in \bar{A} . Then the closure, \bar{L} , of L is strongly Leibniz.*

Proof. It is clear that $\bar{L}(1) = 0$. From Proposition 3.1 we know that \bar{L} is Leibniz. Thus we only need to verify the condition on inverses. Suppose now that $c \in \bar{A}$ and that c is invertible in \bar{A} . If $\bar{L}(c) = \infty$ there is nothing to show, so assume that $c \in \bar{A}^f$. Then there is a sequence $\{a_n\}$ in A that converges to c while $\{L(a_n)\}$ converges to $\bar{L}(c)$ with $L(a_n) \leq \bar{L}(c)$ for all n (so $a_n \in A^f$). Since c is invertible in \bar{A} , and the set of invertible elements of a unital Banach algebra is open, the elements a_n are eventually invertible in \bar{A} . Since A^f is assumed to be spectrally stable in \bar{A} the elements a_n are eventually invertible in A^f . Thus we can adjust the sequence $\{a_n\}$ so that each element is invertible in A^f . Then the sequence $\{a_n^{-1}\}$ converges to c^{-1} , while for each n

$$L(a_n^{-1}) \leq \|a_n^{-1}\|^2 L(a_n) \leq \|a_n^{-1}\|^2 \bar{L}(c).$$

It follows easily that $\bar{L}(c^{-1}) \leq \|c^{-1}\|^2 \bar{L}(c)$. Thus \bar{L} is strongly Leibniz. \square

4. C^* -METRICS

Up to this point we have ignored the crucial analytic property of the seminorms that define quantum metric spaces, i.e., Lip-norms. We recall this property here, for our special context of unital C^* -algebras. Let A be a unital $*$ -algebra equipped with a C^* -norm (but not assumed to be complete). Let L be a seminorm on A such that $L(1) = 0$. Define a metric, ρ_L , on the state space, $S(A)$, of A by

$$\rho_L(\mu, \nu) = \sup\{|\mu(a) - \nu(a)| : a = a^* \text{ and } L(a) \leq 1\}.$$

(Without further hypotheses ρ_L might take the value $+\infty$.) We will say that L is a Lip-norm if the topology on $S(A)$ from ρ_L coincides with the weak- $*$ topology on $S(A)$. In our definition of Lip-norms in definition 2.1 of [28] we, in effect, assumed that our seminorms L were defined only on the self-adjoint part of A , but still defined ρ_L as above. The comments before definition 2.1 of [28] show that if L is a $*$ -seminorm then ρ_L would not change if the condition “ $a = a^*$ ” above were omitted.

We now come to the definition that seems to be dictated by our investigation of vector bundles and Gromov–Hausdorff distance, both for ordinary metric spaces [31] and for quantum ones. It should be viewed as tentative, since future experience may require additional hypotheses.

Definition 4.1. Let A be a unital C^* -normed algebra and let L be a seminorm on A (possibly taking value $+\infty$). We will say that L is a C^* -metric on A if

- a) L is a lower-semicontinuous strongly-Leibniz $*$ -seminorm,
- b) L (restricted to A^{sa}) is a Lip-norm,
- c) A^f is spectrally stable in the completion, \bar{A} , of A .

By a *compact C^* -metric space* we mean a pair (A, L) consisting of a unital C^* -normed algebra A and a C^* -metric L on A .

In using the word “space” above, we should logically be referring to objects in the dual to the category of unital C^* -algebras. But we will not make this distinction explicit during our discussions in this paper.

We need condition c) in Definition 4.1 so that we can apply Proposition 3.2 to conclude that the closure of a C^* -metric is strongly Leibniz and itself satisfies condition c). Hanfeng Li has pointed out to me that the subalgebra of polynomials in the algebra of continuous functions on the unit interval with the usual Lipschitz seminorm shows that condition c) is independent of conditions a) and b).

At this time it is not clear to me how best to define C^* -metric spaces that are locally compact but not compact, though some substantial indications can be gleaned from the results in [15].

Recall [4] that a $*$ -subalgebra B of a C^* -algebra A is said to be stable under the C^2 -function calculus for self-adjoint elements if for any $b \in B$ with $b^* = b$ and any twice continuously differentiable function f on \mathbb{R} , the element $f(b)$ of A , defined by the continuous-function calculus on self-adjoint elements of A , is in fact again in B .

Proposition 4.2. *Every C^* -metric on a unital C^* -normed algebra is semifinite. Let L be a C^* -metric on a unital C^* -normed algebra A , and let \bar{L} be its closure on the completion \bar{A} of A (so \bar{L} is an extension of*

L). Then \bar{L} is a C^* -metric. Let \bar{A}^f be defined as earlier (so now \bar{A}^f is dense in \bar{A}). Then \bar{A}^f is stable both under the holomorphic-function calculus of \bar{A} and the C^2 -calculus on self-adjoint elements of \bar{A} .

Proof. Let L be a C^* -metric on a unital C^* -normed algebra A , and let A^f be defined as above. Suppose that A^f is not dense in A . Then it is easily seen that there is an $a \in A$ with $a^* = a$ that is not in the closure of A^f . By the Hahn–Banach theorem there is a linear functional of norm 1 on the self-adjoint part of A that has value 0 on all of the self-adjoint part of A^f . From lemma 2.1 of [25] it then follows that there are two distinct states of A which agree on A^f . Then the distance between these two states for the metric ρ_L determined by L is 0, which contradicts the requirement that the topology on $S(A)$ determined by ρ_L coincides with the weak-* topology.

The fact that \bar{L} is a C^* -metric is seen as follows. By definition, \bar{L} is closed so lower semi-continuous. As remarked above, \bar{L} is strongly Leibniz by condition c) and Proposition 3.2. The closure of a Lip-norm is again a Lip-norm, giving the same metric on the state-space, as seen in proposition 4.4 of [25]. That \bar{L} satisfies condition c) follows from Proposition 3.1.

The fact that \bar{A}^f is stable under the holomorphic-function calculus of \bar{A} follows immediately from the semifiniteness of \bar{L} and Proposition 3.1. The fact that \bar{A}^f is stable for the C^2 -function calculus on self-adjoint elements of \bar{A} follows quickly from proposition 6.4 of [4], which actually gives a slightly stronger fact. \square

The condition that L be a Lip-norm is often a difficult one to verify for various specific examples. But most of the Lip-norms that have been constructed on C^* -algebras so far are in fact C^* -metrics. We explain this now for several of the classes of examples described in sections 2 and 3 of [27].

Example 4.3. Let A be a unital C^* -algebra, let G be a compact group, and let α be an action of G on A that is ergodic in the sense that if an $a \in A$ satisfies $\alpha_x(a) = a$ for all $x \in G$ then $a \in \mathbb{C}1_A$. Let ℓ be a continuous length function on G , and define a seminorm L on A , as in Example 2.5, by

$$L(a) = \sup\{\|\alpha_x(a) - a\|/\ell(x) : x \notin e_G\}.$$

It is shown in [27] that L is a Lip-norm. But we saw in Example 2.5 that L is lower semicontinuous and strongly Leibniz. Since L is defined on all of A , the spectral stability of \bar{A}^f in A follows from Proposition 3.1.

The next several examples involve “Dirac” operators in various settings.

Example 4.4. This class of examples is the main class discussed in Connes’s first paper [5] on metric aspects of non-commutative geometry. It is discussed briefly as example 3.6 of [27]. Let G be a discrete group and let $A = C_r^*(G)$ be its reduced group C^* -algebra acting on $\ell^2(G)$. Let ℓ be a length function on G . As Dirac operator take the operator $D = M_\ell$ of pointwise multiplication by ℓ on $\ell^2(G)$. The one-parameter unitary group generated by D simply sends t to the operator of pointwise multiplication by the function $e^{it\ell}$. We are then in the context of Examples 2.7 and 2.8. It is easily seen that the dense subalgebra $C_c(G)$ of functions of finite support is in the smooth algebra B^∞ for the action of G on $\mathcal{L}(\ell^2(G))$. As in Example 2.8 we thus obtain a lower-semicontinuous strongly-Leibniz semifinite $*$ -seminorm on A , which for any $f \in C_c(G)$ is given by

$$L(f) = \|[D, f]\|.$$

From Proposition 3.1 it follows that A^f (for the closure of L) is stable for the holomorphic-function calculus. But for stupid length functions L can fail to be a Lip-norm, and it is not easy to see when it is a Lip-norm, and thus a C^* -metric. In [26], by means of a long and interesting argument, it is shown that L is a Lip-norm, and thus a C^* -metric, for $G = \mathbb{Z}^d$ (and even for the twisted group algebra $C^*(\mathbb{Z}^d, \gamma)$ where γ is a bicharacter on \mathbb{Z}^d) when ℓ is either a word-length function or the restriction to \mathbb{Z}^d of a norm on \mathbb{R}^d . In [21] it is shown, by techniques entirely different from those used for the case of \mathbb{Z}^d , that if G is a hyperbolic group and ℓ is a word-length function on G then L is a Lip-norm, and thus a C^* -metric. For other classes of infinite discrete groups, e.g., nilpotent ones, it remains a mystery as to whether L is a Lip-norm if ℓ is a word-length function. Some related examples can be found in [1].

Example 4.5. Let α be an action of the d -dimensional torus \mathbb{T}^d , $d \geq 2$, on a unital C^* -algebra A . In [23] it is shown that for any skew-symmetric real $d \times d$ matrix θ one can deform the product on A to get a new C^* -algebra, A_θ . Connes and Landi [7] show that when M is a compact spin Riemannian manifold and α is a smooth action of \mathbb{T}^d on M , so on $A = C(M)$, leaving the Riemannian metric invariant, and lifting to the spin bundle, then there is a natural Dirac operator for the (usually non-commutative) deformed algebra A_θ . As in Examples 2.8 and 4.3, this Dirac operator determines a $*$ -seminorm, L , on A_θ which

is lower semicontinuous, strongly Leibniz, and semifinite. Hanfeng Li [16] showed that L is a Lip^* -norm. Thus L is a C^* -metric.

5. QUOTIENT SEMINORMS AND PROXIMITY

We now try to modify the definition of quantum Gromov–Hausdorff distance so as to use the above definition of C^* -metrics. This involves quotient seminorms, so we begin by exploring them. There are at least three difficulties that confront us, namely that the quotient of a Leibniz seminorm may not be Leibniz, that the quotient of a strongly Leibniz seminorm, even if it is Leibniz, may not be strongly Leibniz, and that reasonable $*$ -seminorms can agree on self-adjoint elements but still be distinct. We begin by considering the first difficulty.

Let L be a Leibniz seminorm on a unital normed algebra C , and let $\pi : C \rightarrow A$ be a unital homomorphism from C onto a unital normed algebra A . Let \tilde{L}^A be the quotient seminorm on A , defined by

$$\tilde{L}^A(a) = \inf\{L(c) : c \in C \text{ and } \pi(c) = a\}.$$

It is known [4] that \tilde{L}^A need not be Leibniz. (See also lemma 4.3 of [18] and the comments just before it.) But the situation can be partly rescued by the following definition.

Definition 5.1. Let C , A , π and L be as above, and assume that π is norm non-increasing. We say that L is π -compatible if for every $a \in A$ and every $\varepsilon > 0$ there is a $c \in C$ such that $\pi(c) = a$ and simultaneously

$$L(c) \leq \tilde{L}^A(a) + \varepsilon \quad \text{and} \quad \|c\| \leq \|a\| + \varepsilon.$$

Proposition 5.2. Let C , A , π and L be as above. If L is π -compatible then the norm on A coincides with the quotient norm from C , and \tilde{L}^A is Leibniz.

Proof. The statement about the norms is easily verified. Suppose now that $a, b \in A$ and $\varepsilon > 0$ are given. Since L is π -compatible, we can find $c, d \in C$ such that $\pi(c) = a$ and $\pi(d) = b$ and the conditions of Definition 5.1 are satisfied. Then $\pi(cd) = \pi(ab)$, and so

$$\begin{aligned} \tilde{L}^A(ab) &\leq L(cd) \leq L(c)\|d\| + \|c\|L(d) \\ &\leq (\tilde{L}^A(a) + \varepsilon)(\|b\| + \varepsilon) + (\|a\| + \varepsilon)(\tilde{L}^A(b) + \varepsilon). \end{aligned}$$

Since ε is arbitrary, we see that \tilde{L}^A is Leibniz. \square

However, I do not know of a useful way to partly rescue the difficulty that if L is strongly Leibniz and \tilde{L}^A is Leibniz there seems to be no reason that \tilde{L}^A need be strongly Leibniz (though I do not have an example showing this difficulty).

We now consider the third difficulty. It is quite instructive to first consider ordinary metric spaces. For this purpose π -compatibility is useful.

Proposition 5.3. *Let (Z, ρ) be a compact metric space, and let $C = C(Z)$ be its C^* -algebra of continuous complex-valued functions. Let X be a closed subset of Z , let $A = C(X)$, and let $\pi : C \rightarrow A$ be the usual restriction homomorphism. Then the Leibniz seminorm L^ρ for ρ is π -compatible.*

Proof. Let $f \in A$. Let Q be the radial retraction of \mathbb{C} onto its ball of radius $\|f\|_\infty$ centered at 0. It is easily seen that the Lipschitz constant of Q is 1. Then for any $h \in C$ with $\pi(h) = f$ we can set $g = Q \circ h$ and we will have $\pi(g) = f$ and $L(g) \leq L(h)$ while $\|g\| = \|f\|$. This quickly gives the desired result. \square

We remark that the above argument does not work for matrix-valued functions, as employed in [31], since the radial retraction no longer has Lipschitz constant 1 [30].

While Proposition 5.3 appears favorable, the difficulty is that the quotient of L^ρ on A need not agree with the Lipschitz seminorm from the metric ρ_X on X coming from restricting ρ :

Example 5.4. (See [37, 30].) Let (X, ρ_X) be the metric space containing exactly 3 points, at distance 2 from each other. We can ask what the Gromov–Hausdorff distance is from (X, ρ_X) to a metric space consisting of one point, say p . It is easily seen that the answer is 1, with the metric ρ on $Z = X \cup \{p\}$ that extends ρ_X giving p distance 1 to each point of X . Now let f be the function on X which sends the three points of X to the three different cube roots of 1 in \mathbb{C} . It is not difficult to see that the extension of f to Z that has the smallest Lipschitz norm is the extension g that sends p to 0. But $L^\rho(g)$ is easily seen to be substantially larger than $L^{\rho_X}(f)$. As remarked in [31, 30], this is possible because the metric on Z is somewhat hyperbolic.

On the other hand, for any compact metric space (Z, ρ) , any closed subset X of Z , and for any $f \in C_{\mathbb{R}}(X)$, there is a $g \in C_{\mathbb{R}}(Z)$ with $g|_X = f$, $\|g\| = \|f\|$ and $L^\rho(g) = L^{\rho_X}(f)$ [30]. This shows in particular that here L^{ρ_X} does coincide with the quotient seminorm from L^ρ . It also means that for the situation of Example 5.4 we have two Leibniz seminorms on $C(X)$ which agree on real-valued functions but are nevertheless distinct. (For a related phenomenon see [22].) From the comments at the end of the first paragraph of Section 4 we see that these two seminorms will give the same metrics on the set of probability measures on X , and in particular the same metrics on X .

We now turn our attention to Gromov–Hausdorff distance. Let (A, L_A) and (B, L_B) be C^* -metric spaces. The evident way to adapt the definition of quantum Gromov–Hausdorff distance given in definition 4.2 of [28] is to require that the seminorms L considered on $A \oplus B$ be C^* -metrics. Example 5.4 shows that we can not require the quotient of L on A to agree with L_A , except on self-adjoint elements (though for the main class of examples considered in later sections they will agree, so those examples are better-behaved than Example 5.4). Then we do not know whether the quotient is Leibniz. We could impose π -compatibility to ensure this, but then we still may not have the strong Leibniz property, so it is not clear that it is useful to impose this.

Perhaps as our topic develops in the future it will become clearer what are the best conditions to impose. Anyway, guided by the above observations, we set, parallel to notation 4.1 of [28]:

Notation 5.5. Let (A, L_A) and (B, L_B) be compact C^* -metric spaces. We let $\mathcal{M}_C(L_A, L_B)$ denote the collection of all C^* -metrics, L , on $A \oplus B$ such that the quotient of L on A agrees with L_A on self-adjoint elements of A , and similarly for the quotient of L on B .

We want to modify the definition of quantum Gromov–Hausdorff distance, dist_q , given in definition 4.2 of [28] by requiring that the seminorms involved there are in $\mathcal{M}_C(L_A, L_B)$. But I am not able to show that the resulting notion satisfies the triangle inequality. When one tries to imitate the proof of the triangle inequality for dist_q given in theorem 4.3 of [28], one of the main obstacles is in showing that the Lip-norm L_{AC} of lemma 4.6, which is defined as a quotient seminorm, is a C^* -metric. I would not be surprised if the triangle inequality fails. So the term “distance” should not be used. I will use instead the term “proximity”. Thus:

Definition 5.6. Let (A, L_A) and (B, L_B) be compact C^* -metric spaces. We define their proximity by

$$\text{prox}(A, B) = \inf\{\text{dist}_H^{\rho_L}(S(A), S(B)) : L \in \mathcal{M}_C(L_A, L_B)\}.$$

This definition makes sense in the following way. Both $S(A)$ and $S(B)$ are closed subsets of $S(A \oplus B)$. Much as at the beginning of Section 4, ρ_L is a metric on $S(A \oplus B)$, and $\text{dist}_H^{\rho_L}$ is ordinary Hausdorff distance with respect to ρ_L . We note that the hypotheses in the definition of $\mathcal{M}_C(L_A, L_B)$ are such that proposition 3.1 of [28] applies, so that for any $L \in \mathcal{M}_C(L_A, L_B)$ the restrictions of ρ_L to $S(A)$ and $S(B)$ coincide with ρ_{L_A} and ρ_{L_B} . Put another way, when we associate to each $L \in \mathcal{M}_C(L_A, L_B)$ its restriction to the self-adjoint part of $A \oplus B$ we obtain a map from $\mathcal{M}_C(L_A, L_B)$ to $\mathcal{M}(L_A^s, L_B^s)$, where L_A^s denotes

the restriction of L_A to the self-adjoint part of A , and similarly for L_B^s . This map need not be either injective or surjective.

It is clear that

$$\text{dist}_q(A, B) \leq \text{prox}(A, B),$$

since $\text{prox}(A, B)$ is an infimum over a subset of the seminorms used to define $\text{dist}_q(A, B)$. Thus if we have a sequence (B^n, L_{B^n}) of C^* -metric spaces for which the sequence $\text{prox}(A, B^n)$ converges to 0, then it follows that (B^n, L_{B^n}) converges to (A, L_A) for quantum Gromov–Hausdorff distance. For this reason the absence of the triangle inequality will not be too serious a problem. The advantage of prox , as mentioned earlier, is that the use of seminorms L on $A \oplus B$ that are C^* -metrics permits one to try to generalize to C^* -metric spaces the results about vector bundles obtained in [31] for ordinary metric spaces. (We plan to discuss this in a future paper.)

6. BIMODULE-BRIDGES

In the development of quantum Gromov–Hausdorff distance given in [28] and used in [29], a very convenient method for constructing suitable seminorms L on $A \oplus B$ involved suitable continuous seminorms N on $A \oplus B$ that we called “bridges”, with L then defined as

$$L(a, b) = L_A(a) \vee L_B(b) \vee N(a, b).$$

Within the context of the present paper it is natural to require that N satisfy a suitable Leibniz condition. There is an evident condition to consider, coming from viewing N as a seminorm on the algebra $A \oplus B$. But it seems more appropriate to require the stronger condition

$$N((a, b)(a', b')) \leq N(a, b)\|b'\| + \|a\|N(a', b').$$

Examples show that this condition can be interpreted as indicating that N only provides metric data between A and B , and not within A or within B .

We will find it very useful to use bridges that come from normed bimodules. Such bridges will satisfy the Leibniz condition stated above. Let A and B be unital C^* -algebras, and let Ω be an A - B -bimodule. We say that Ω is a normed bimodule if it is equipped with a norm that satisfies, much as in Section 2,

$$\|a\omega b\| \leq \|a\|\|\omega\|\|b\|$$

for all $a \in A$, $b \in B$ and $\omega \in \Omega$. We assume that the identity elements of A and B both act as the identity operator on Ω .

Definition 6.1. Let (A, L_A) and (B, L_B) be C^* -metric spaces. By a *bimodule bridge* for (A, L_A) and (B, L_B) we mean a normed A - B -bimodule Ω together with a distinguished element $\omega_0 \neq 0$ such that when we form the seminorm N on $A \oplus B$ defined by

$$N(a, b) = \|a\omega_0 - \omega_0 b\|,$$

it has the property that for any $a \in A$ with $a = a^*$ and any $\varepsilon > 0$ there is a $b \in B$ with $b^* = b$ such that

$$L_B(b) \vee N(a, b) \leq L_A(a) + \varepsilon,$$

and similarly for A and B interchanged.

Theorem 6.2. Let (Ω, ω_0) be a bimodule bridge for the C^* -metric spaces (A, L_A) and (B, L_B) , and let N be defined as above in terms of (Ω, ω_0) . Define L on $A \oplus B$ by

$$L(a, b) = L_A(a) \vee L_B(b) \vee N(a, b) \vee N(a^*, b^*).$$

Then $L \in \mathcal{M}_C(L_A, L_B)$.

Proof. One can show directly that N is strongly Leibniz, or view Ω as an $(A \oplus B)$ -bimodule in the evident way and apply Proposition 2.1. Since N is also continuous, it follows from Proposition 1.2 that L is lower semicontinuous and strongly Leibniz. Clearly L is a $*$ -seminorm. Thus condition a) of Definition 4.1 is satisfied.

We now want to apply theorem 5.2 of [28] to show that L is a Lip^* -norm. We must thus show that $N \vee N^*$, restricted to the self-adjoint part of $A \oplus B$ is a bridge as defined in definition 5.1 of [28]. From its bimodule source it is clear that $N(1_A, 1_B) = 0$, while $N(1_A, 0) \neq 0$ since $\omega_0 \neq 0$. Since also N is continuous, it follows that the first two conditions of definition 5.1 are satisfied. The main technical condition of Definition 6.1 directly implies that condition 3 of definition 5.1 of [28] is satisfied, so that $N \vee N^*$ is indeed a bridge, and so L , restricted to self-adjoint elements, is a Lip -norm. Thus L is a Lip^* -norm, and so condition b) of Definition 4.1 is satisfied.

Because N is clearly finite, $(A \oplus B)^f$, as defined for L , coincides with $A^f \oplus B^f$. From the fact that A^f and B^f are by assumption spectrally stable in their completion it follows easily that $(A \oplus B)^f$ is spectrally stable in its completion. Thus L satisfies condition c) of Definition 4.1, so that L is a C^* -metric.

Suppose now that we are given $a \in A$ with $a = a^*$. From the formula for L it is clear that $L(a, b) \geq L_A(a)$ for all $b \in B$. Let $\varepsilon > 0$ be given. Then by Definition 6.1 there is a $b \in B$ with $b = b^*$ such that

$$L_B(b) \vee N(a, b) \leq L_A(a) + \varepsilon.$$

Since N and N^* agree on self-adjoint elements, it follows that $L(a, b) \leq L_A(a) + \varepsilon$. Since ε is arbitrary, it follows that the quotient of L on A applied to a gives $L_A(a)$. In the same way the quotient of L on B , restricted to self-adjoint elements, gives L_B on self-adjoint elements. Thus $L \in \mathcal{M}_C(L_A, L_B)$. \square

In the next sections we will see how to construct useful bimodule bridges for “matrix algebras converging to the sphere”.

Hanfeng Li has pointed out to me that prox is dominated by the “nuclear Gromov-Hausdorff distance” dist_{nu} that he defined in remark 5.5 of [18] and studied further in section 5 of [14]. He gives a proof of this in the appendix of [19]. (He uses the term “nuclear” because this distance has favorable properties for nuclear C^* -algebras.) We sketch here how this works, so that it can be easily compared with what we have done above. The crux of Li’s approach is that he restricts attention to bimodules of a quite special kind. Specifically, for unital C^* -algebras A and B let $\mathcal{H}(A, B)$ denote the collection of all triples (D, ι_A, ι_B) consisting of a unital C^* -algebra D and injective (so isometric) unital homomorphisms ι_A and ι_B from A and B into D . We can then view D as an A - B -bimodule in the evident way. For a C^* -metric L_A on A Li sets

$$\mathcal{E}(L_A) = \{a \in A^{sa} : L_A(a) \leq 1\},$$

the L_A -unit-ball in A^{sa} . Then for any $(D, \iota_A, \iota_B) \in \mathcal{H}(A, B)$ he considers

$$\text{dist}_H(\iota_A(\mathcal{E}(L_A)), \iota_B(\mathcal{E}(L_B))),$$

the ordinary Hausdorff distance in D for the norm of D . Even though $\mathcal{E}(L_A)$ and $\mathcal{E}(L_B)$ are unbounded, this distance is finite, for the following reason. Let r_A be the radius of (A, L_A) , as defined in section 2 of [25], so that $\|\tilde{a}\| \leq r_A L_A(a)$ for any $a \in A^{sa}$, where $\tilde{}$ denotes image in the quotient $A^{sa}/\mathbb{R}1_A$, with the quotient norm. Then if $a \in \mathcal{E}(L_A)$ so that $L_A(a) \leq 1$, it follows that $a = a' + t1_A$ for some $t \in \mathbb{R}$ and $a' \in A^{sa}$ with $\|a'\| \leq r_A$. Let $b = t1_B$, so that $b \in \mathcal{E}(L_B)$. Then

$$\|\iota_A(a) - \iota_B(b)\| = \|a'\| \leq r_A.$$

Thus $\iota_A(\mathcal{E}(L_A))$ is in the r_A -neighborhood of $\iota_B(\mathcal{E}(L_B))$. By also interchanging the roles of a and b we see that

$$\text{dist}_H(\iota_A(\mathcal{E}(L_A)), \iota_B(\mathcal{E}(L_B))) \leq \max(r_A, r_B).$$

Then Li defines $\text{dist}_{nu}(A, B)$ (or, more precisely, $\text{dist}_{nu}(L_A, L_B)$) to be

$$\inf\{\text{dist}_H(\iota_A(\mathcal{E}(L_A)), \iota_B(\mathcal{E}(L_B))) : (D, \iota_A, \iota_B) \in \mathcal{H}(A, B)\}.$$

Li shows as follows that dist_{nu} satisfies the triangle inequality. Suppose that a third compact C^* -metric space (C, L_C) is given. Let $d_{AB} =$

$\text{dist}_{nu}(A, B)$, and similarly for d_{BC} and d_{AC} . Given $\varepsilon > 0$ we can find $(D, \iota_A, \iota_B) \in \mathcal{H}(A, B)$ and $(E, \rho_B, \rho_C) \in \mathcal{H}(B, C)$ such that

$$\text{dist}_H(\iota_A(\mathcal{E}(L_A)), \iota_B(\mathcal{E}(L_B))) \leq d_{AB} + \varepsilon,$$

and similarly for d_{BC} . Let $F = D *_B E$ be an amalgamated product of D and E over B (using the inclusions ι_B and ρ_B). This means that there are unital injective homomorphisms σ_D and σ_E of D and E into F such that $\sigma_D \circ \iota_B = \sigma_E \circ \rho_B$. (It is natural to cut down to the subalgebra generated by the images of D and E in F .)

Before continuing, we remark that it is easy to construct a universal amalgamated free product, $A *_C B$, if one does not insist that the homomorphisms into it from A and B are injective. One takes the quotient of the universal (i.e. full) free product $A * B$ by the ideal generated by the desired relations from C . See [20]. What is not as simple is to show that the evident homomorphisms of A and B into the universal $A *_C B$ are injective. This was first shown by Blackadar in [3]. In a comment added in proof in that paper, Blackadar says that John Phillips has shown him a preferable proof. Blackadar has shown me this proof of John Phillips, and since it seems not to have appeared in print up to now, we sketch it here. Hanfeng Li has pointed out to me that a version of the argument in a substantially more complicated situation appears in the proof of proposition 2.2 of [2].

To simplify notation we simply view C as a unital subalgebra of each of A and B . The crux of the matter is to show that there are faithful (non-degenerate) representations of A and B on the same Hilbert space whose restrictions to C are equal. We construct such representations as follows.

- (1) Let (π_1, \mathcal{H}_1) be a faithful representation of A . Form the restricted representation $(\pi_1|_C, \mathcal{H}_1)$ of C , and extend it to a representation $(\rho_1, \mathcal{H}_1 \oplus \mathcal{K}_1)$ of B . (This can be done by decomposing into cyclic representations and extending their states – see lemma 2.1 of [2].)
- (2) Notice that $\rho_1|_C$ carries \mathcal{H}_1 into itself and so carries \mathcal{K}_1 into itself. Extend $(\rho_1|_C, \mathcal{K}_1)$ to a representation $(\pi_2, \mathcal{K}_1 \oplus \mathcal{H}_2)$ of A .
- (3) Extend $(\pi_2|_C, \mathcal{H}_2)$ to a representation $(\rho_2, \mathcal{H}_2 \oplus \mathcal{K}_2)$ of B .
- (4) Continue this process through all the positive integers, and form $\mathcal{H} = \bigoplus_1^\infty (\mathcal{H}_j \oplus \mathcal{K}_j)$. The π_j 's and ρ_j 's combine to give representations π and ρ of A and B on \mathcal{H} which can be checked to agree on C . Since π_1 was chosen to be a faithful representation of A , so is π . Thus the homomorphism from A into $A *_C B$ must be injective. The situation is symmetric for A and B , so the homomorphism from B into $A *_C B$ must also be injective.

We return to demonstrating the triangle inequality for dist_{nu} . Let $\tau_A = \sigma_D \circ \iota_A$ and $\tau_C = \sigma_E \circ \rho_C$. Then $(F, \tau_A, \tau_C) \in \mathcal{H}(A, C)$. Furthermore, if $a \in \mathcal{E}(L_A)$ then there is a $b \in \mathcal{E}(L_B)$ such that $\|\iota_A(a) - \iota_B(b)\| \leq d_{AB} + \varepsilon$, so that $\|\tau_A(a) - \sigma_D(\iota_B(b))\| \leq d_{AB} + \varepsilon$. In the same way there exists $c \in \mathcal{E}(L_C)$ such that $\|\sigma_E(\rho_B(b)) - \tau_C(c)\| \leq d_{BC} + \varepsilon$. But $\sigma_D(\iota_B(b)) = \sigma_E(\rho_B(b))$, and so

$$\|\tau_A(a) - \tau_C(c)\| \leq d_{AB} + d_{BC} + 2\varepsilon.$$

In this way we find that

$$\text{dist}_{nu}(L_A, L_C) \leq \text{dist}_{nu}(L_A, L_B) + \text{dist}_{nu}(L_B, L_C).$$

Further favorable properties of dist_{nu} are presented in [18, 14] that we will not discuss here.

Given $(D, \iota_A, \iota_B) \in \mathcal{H}(A, B)$, we can view D as a normed A - B -bimodule in the evident way, and as special element we can choose $\omega_0 = 1_D$. The corresponding bounded seminorm N_D on $A \oplus B$ is then simply defined by

$$N_D(a, b) = \|\iota_A(a) - \iota_B(b)\|.$$

Given C^* -metrics L_A and L_B on A and B , we can seek constants γ such that $\gamma^{-1}N_D$ is a bimodule bridge for L_A and L_B . Let $\delta = \text{dist}_H(\iota_A(\mathcal{E}(L_A)), \iota_B(\mathcal{E}(L_B)))$. Given any $\varepsilon > 0$ we show that $\delta + \varepsilon$ is such a constant. Let $a \in A^{sa}$ with $L_A(a) = 1$. Then there is a $b \in B^{sa}$ such that $L_B(b) \leq 1$ and $\|\iota_A(a) - \iota_B(b)\| \leq \delta + \varepsilon$, so that

$$L_B(b) \vee (\delta + \varepsilon)^{-1}N_D(a, b) \leq 1 = L_A(a).$$

We can interchange the roles of A and B . Thus we see that $(\delta + \varepsilon)^{-1}N_D$ is indeed a bimodule bridge. Notice that for any $a \in A$ and $b \in B$ we have $N(a^*, b^*) = N(a, b)$. Thus when we define L on $A \oplus B$ by

$$L(a, b) = L_A(a) \vee L_B(b) \vee (\delta + \varepsilon)^{-1}N_D(a, b)$$

it follows from Theorem 6.2 that $L \in \mathcal{M}_C(L_A, L_B)$.

But even more is true. As suggested by Li, we will follow the argument in the last paragraph of the proof of proposition 4.7 of [17]. Let $\mu \in S(A)$. View A and B as subalgebras of D via ι_A and ι_B . By the Hahn-Banach theorem, extend μ to a state $\tilde{\nu}$ of D , and then restrict $\tilde{\nu}$ to B to get $\nu \in S(B)$. Then for $a \in A^{sa}$ and $b \in B^{sa}$ we have

$$|\mu(a) - \nu(b)| = |\tilde{\nu}(a) - \tilde{\nu}(b)| \leq \|\iota_A(a) - \iota_B(b)\| \leq (\delta + \varepsilon)L(a, b),$$

where L is defined as above. Consequently if $L(a, b) \leq 1$ then we have $|\mu(a) - \nu(b)| \leq \delta + \varepsilon$. Thus μ is in the $\delta + \varepsilon$ -neighborhood of $S(B)$ for

the metric ρ_L on $S(A \oplus B)$. The same argument works with the roles of A and B reversed. Since ε is arbitrary, we see from this that

$$\text{prox}(A, B) \leq \text{dist}_{nu}(A, B),$$

as asserted.

In [18, 14] Li indicates that dist_{nu} works very well with many of the classes of specific examples whose metric aspects have been studied. In particular, he pointed out to me that dist_{nu} can be used to give an alternate proof of our Main Theorem (in a qualitative way). This alternate proof is attractive because of its quite general approach. However, a proof via dist_{nu} appears to me to be less concrete and quantitative than that which we give in the next sections, both because the proof via dist_{nu} uses a somewhat deep theorem of Blanchard on the subtrivialization of continuous fields of nuclear C^* -algebras (as discussed in remark 5.5 of [18]), and because of its use of the Hahn-Banach theorem seen just above. The proof we will give provides specific estimates for the approximation, and provides a constructive way of finding a state for one of the algebras that is close to a given state of the other algebra.

7. MATRIX ALGEBRAS AND HOMOGENEOUS SPACES

In this section we begin the study of our main example. Our discussion will be fairly parallel to that in [29] but with some important differences. For the reader's convenience we will include here some fragments of [29] in order to make precise our setting. We will usually use the notation used in [29].

Let G be a compact group (perhaps even finite at first). Let U be an irreducible unitary representation of G on a Hilbert space \mathcal{H} . Let $B = \mathcal{L}(\mathcal{H})$ denote the C^* -algebra of linear operators on \mathcal{H} (a "full matrix algebra"). There is a natural action, α , of G on B by conjugation by U . That is, $\alpha_x(T) = U_x T U_x^*$ for $x \in G$ and $T \in B$. We introduce metric data into the picture by choosing a continuous length-function, ℓ , on G . We require that ℓ satisfy the additional condition that $\ell(xyx^{-1}) = \ell(y)$ for $x, y \in G$. This ensures that the metric on G defined by ℓ is invariant under both left and right translations. As in Example 2.5 we define a seminorm, L_B , on B by

$$L_B(T) = \sup\{\|\alpha_x(T) - T\|/\ell(x) : x \neq e_G\}.$$

Then L_B is a C^* -metric on B for the reasons given in Example 4.3.

Let P be a rank-one projection in B . Let $H = \{x \in G : \alpha_x(P) = P\}$, the stability subgroup for P . Let $A = C(G/H)$, the C^* -algebra of continuous complex-valued functions on G/H . We let λ denote the

usual action of G on G/H , and so on A , by translation. We define a seminorm, L_A , on A as in Example 2.5 by

$$L_A(f) = \sup\{\|\lambda_x(f) - f\|/\ell(x) : x \neq e_G\}.$$

Again, L_A is a C^* -metric for the reasons given in Example 4.3.

We can then ask for estimates of $\text{prox}(A, B)$. To obtain such an estimate we need to construct a suitable C^* -metric on $A \oplus B$. We do this as follows. For any $T \in B$ its Berezin covariant symbol, σ_T , is defined by

$$\sigma_T(x) = \text{tr}(T\alpha_x(P)),$$

for $x \in G$. Here tr is the usual unnormalized trace on B . Because of the definition of H we see that $\sigma_T \in C(G/H) = A$. When the $\alpha_x(P)$'s are viewed as giving states of B via tr as above, they form a ‘‘coherent state’’, assigning a pure state of B to each pure state of A . Once we note that tr is α -invariant, it is easy to see that σ is a unital, positive, norm-non-increasing α - λ -equivariant operator from B to A . However eventually one really wants also the property that if $\sigma_T = 0$ then $T = 0$. This is equivalent to the linear span of the $\alpha_x(P)$'s in B being all of B . It is an interesting question as to which representations U admit such a P , and how many such P 's, even for finite groups.

We let $\Omega = \mathcal{L}(B, A)$, the Banach space of linear operators from B to A , equipped with the operator norm corresponding to the C^* -norms on A and B . (Perhaps we should be using the space of completely bounded operators here.) We let M and Λ denote the left regular representations of A and B . Then Ω is an A - B -bimodule for the operations

$$f\omega = M_f \circ \omega \quad \text{and} \quad \omega T = \omega \circ \Lambda_T.$$

It is easily checked that Ω is a normed A - B -bimodule. Of course $\sigma \in \Omega$. We will take our bimodule bridge for (A, L_A) and (B, L_B) to be of the form $(\Omega, \gamma^{-1}\sigma)$ where γ is a positive real number that is yet to be determined. Set

$$N_\sigma(f, T) = \|M_f \circ \sigma - \sigma \circ \Lambda_T\|.$$

Then the seminorm N from $(\Omega, \gamma^{-1}\sigma)$ is defined by

$$N(f, T) = \gamma^{-1}N_\sigma(f, T).$$

We need to determine the values of γ for which $(\Omega, \gamma^{-1}\sigma)$ is a bimodule bridge so that, in particular, the corresponding seminorm L has L_A and L_B as quotients for self-adjoint elements. But, as a first step in showing what the implication for proximity will be, we have:

Proposition 7.1. *Suppose that γ is such that $(\Omega, \gamma^{-1}\sigma)$ is a bimodule bridge for L_A and L_B , and let N be the seminorm it determines. Let*

$L = L_A \vee L_B \vee N \vee N^*$ and let ρ_L be the metric on $S(A \oplus B)$ that L determines. Then $S(A)$ is in the γ -neighborhood of $S(B)$ for ρ_L .

Proof. Let $\mu \in S(A)$. We must find a $\nu \in S(B)$ such that $\rho_L(\mu, \nu) \leq \gamma$. We choose $\nu = \mu \circ \sigma$. Let $(f, T) \in A \oplus B$ be such that $L(f, T) \leq 1$, so that $N(f, T) \leq 1$ and thus $N_\sigma(f, T) \leq \gamma$. Then

$$\begin{aligned} |\mu(f, T) - \nu(f, T)| &= |\mu(f) - \mu(\sigma_T)| \leq \|f - \sigma_T\| \\ &= \|(M_f \circ \sigma - \sigma \circ \Lambda_T)(I)\| \leq N_\sigma(f, T) \leq \gamma, \end{aligned}$$

where I is the identity element in B . From the definition of ρ_L it follows that $\rho_L(\mu, \nu) \leq \gamma$. \square

We remark that in our earlier paper on “matrix algebras converge to the sphere” [29] the bridge N that we had used was $N(f, T) = \gamma^{-1}\|f - \sigma_T\|$. The above calculation reveals that this old N is related to our new one just by applying our $M_f \circ \sigma - \sigma \circ \Lambda_T$ to the identity operator. The old N is not Leibniz.

To proceed further we now obtain another expression for N_σ which will be more convenient for some purposes. We note that for $S, T \in B$ and $f \in A$ we have

$$(M_f \circ \sigma - \sigma \circ \Lambda_T)(S) = f\sigma_S - \sigma_{TS},$$

and that when this is evaluated at $x \in G/H$ we obtain

$$\begin{aligned} f(x)\sigma_S(x) - \sigma_{TS}(x) &= f(x)\operatorname{tr}(S\alpha_x(P)) - \operatorname{tr}(TS\alpha_x(P)) \\ &= \operatorname{tr}(\alpha_x(P)(f(x)I - T)S). \end{aligned}$$

The operator norm of $M_f \circ \sigma - \sigma \circ \Lambda_T$ is then the supremum of the absolute value of the above expression taken over all $x \in G/H$ and $S \in B$ with $\|S\| \leq 1$. But tr gives a pairing that expresses the dual of B with its operator norm as B with the trace-class norm, which we denote by $\|\cdot\|_1$. From this fact we see that

$$\|M_f \circ \sigma - \sigma \circ \Lambda_T\| = \sup\{\|\alpha_x(P)(f(x)I - T)\|_1 : x \in G/H\}.$$

But if R is a rank-one operator then $R^*R = r^2Q$ for some rank-one projection Q and some $r \in \mathbb{R}^+$, so that

$$\|R\|_1 = \operatorname{tr}((R^*R)^{1/2}) = r = \|R^*R\|^{1/2} = \|R\|,$$

where the norm on the right is the operator norm. In this way we obtain:

Proposition 7.2. *For $f \in A$ and $T \in B$ we have*

$$N_\sigma(f, T) = \sup\{N_x(f, T) : x \in G/H\}$$

where $N_x(f, T) = \|\alpha_x(P)(f(x)I - T)\|$.

We remark that $N_x(f, T)$ can easily be checked to be strongly Leibniz.

8. THE CHOICE OF THE CONSTANT γ

Let us first see what choices of γ ensure that L has L_A as a quotient. It suffices to choose γ such that for any $f \in A$ we can find $T \in B$ such that $L_B(T) \vee N(f, T) \leq L_A(f)$. On G/H let us momentarily use the G -invariant measure of mass 1 to give A the norm from $L^2(G/H)$. Similarly, on B we put the Hilbert–Schmidt norm from the *normalized* trace. Then σ has an adjoint operator, which we denote by $\check{\sigma}$. It is easily computed [29] to be defined by

$$\check{\sigma}_f = d \int_{G/H} f(x) \alpha_x(P) dx,$$

where d is the dimension of \mathcal{H} . One can easily verify that $\check{\sigma}$ is a positive and λ - α -equivariant map from A to B . Furthermore, $\check{\sigma}_1 = d \int \alpha_x(P) dx$, which is clearly α -invariant, and so is a scalar multiple of I since U is irreducible. But clearly the usual trace of $d \int \alpha_x(P) dx$ is d . Thus $\check{\sigma}_1 = I$, that is, $\check{\sigma}$ is unital. (This is why we used the normalized traces in defining $\check{\sigma}$.) It follows that $\check{\sigma}$ is also norm non-increasing.

Then, given $f \in A$, we will choose T to be $T = \check{\sigma}_f$. It is easily seen (as in the proof of proposition 1.1 of [29]) that $L_B(\check{\sigma}_f) \leq L_A(f)$. For any $x \in G/H$ we have by equivariance of $\check{\sigma}$

$$N_x(f, \check{\sigma}_f) = \|\alpha_x(P)(f(x)I - \check{\sigma}_f)\| = \|P((\lambda_x^{-1}f)(e)I - \check{\sigma}_{\lambda_x^{-1}f})\|.$$

Since f is arbitrary and L_A is λ -invariant, it suffices for us to consider $\|P(f(e)I - \check{\sigma}_f)\|$. But

$$\begin{aligned} \|P(f(e)I - \check{\sigma}_f)\| &= \left\| P \left(f(e)d \int \alpha_y(P) dy - d \int f(y) \alpha_y(P) dy \right) \right\| \\ &= d \left\| \int (f(e) - f(y)) P \alpha_y(P) dy \right\| \\ &\leq L_A(f) d \int \rho_{G/H}(e, y) \|P \alpha_y(P)\| dy, \end{aligned}$$

where $\rho_{G/H}$ is the ordinary metric on G/H from L_A . From all of this we obtain:

Proposition 8.1. *Set $\gamma^A = d \int \rho_{G/H}(e, y) \|P \alpha_y(P)\| dy$. Then for any $\gamma \geq \gamma^A$ the seminorm $L = L_A \vee L_B \vee \gamma^{-1}(N_\sigma \vee N_\sigma^*)$ on $A \oplus B$ has L_A as its quotient on A .*

We remark that in the above proposition we do not have to restrict attention to self-adjoint elements, in contrast to the requirement in Definition 6.1. Note that $\check{\sigma}_f = (\check{\sigma}_f)^*$. I do not know whether the above condition on γ is the best that can be obtained in the absence of further hypotheses on G , U , P and ℓ .

We now consider the quotient of L on B . Given $T \in B$ we seek $f \in A$ such that $L_A(f) \vee N(f, T) \leq L_B(T)$. We choose $f = \sigma_T$, and seek what requirement this puts on γ . As above, it is easy to check that $L_A(\sigma_T) \leq L_B(T)$. Again by equivariance we have

$$N_x(\sigma_T, T) = \|\alpha_x(P)(\text{tr}(T\alpha_x(P))I - T)\| = \|P(\text{tr}(P\alpha_x^{-1}(T))I - \alpha_x^{-1}(T))\|.$$

Since T is arbitrary and L_B is α -invariant, it suffices to choose γ large enough that $\|P\text{tr}(PT) - PT\| \leq \gamma L_B(T)$ for all $T \in B$. Notice that the left-hand side gives a seminorm (with value 0 for $T = P$ or I) on the quotient space $\tilde{B} = B/\mathbb{C}I$, while L_B gives a norm on \tilde{B} . Since B is finite-dimensional, there does exist a finite γ such that the above inequality is satisfied. Notice also that $\sigma_{T^*} = (\sigma_T)^-$. Thus we obtain:

Proposition 8.2. *Define γ^B by*

$$\gamma^B = \sup\{\|P\text{tr}(PT) - PT\| : T \in B \text{ and } L_B(T) \leq 1\}.$$

Then γ^B is finite, and for any $\gamma \geq \gamma^B$ the seminorm $L = L_A \vee L_B \vee \gamma^{-1}(N_\sigma \vee N_\sigma^)$ on $A \oplus B$ has L_B as its quotient on B .*

For later use we now express $\|P(\text{tr}(PT)I - T)\|$ in a different form. Since taking adjoints is an isometry, and by the C^* -relation, and by the fact that if R is a positive operator then $\|PRP\| = \text{tr}(PRP)$ because P is of rank 1, we have

$$\begin{aligned} \|P(\text{tr}(PT)I - T)\|^2 &= \|P(\text{tr}(PT)I - T)(\text{tr}(PT)I - T)^*P\| \\ &= \text{tr}\left(P(\text{tr}(PT)I - T)(\text{tr}(PT)I - T)^*P\right) \\ &= |\text{tr}(PT)|^2 - \text{tr}(PTP)\overline{\text{tr}(PT)} \\ &\quad - \text{tr}(PT)\text{tr}(PT^*P) + \text{tr}(PTT^*P) \\ &= \text{tr}(PTT^*P) - |\text{tr}(PT)|^2. \end{aligned}$$

Thus:

Proposition 8.3. *For any $T \in B$ we have*

$$\|P(\text{tr}(PT)I - T)\| = (\text{tr}(PTT^*P) - |\text{tr}(PT)|^2)^{1/2}.$$

We remark that if ξ is a unit vector in the range of P then

$$\text{tr}(PTT^*P) - |\text{tr}(PT)|^2 = \langle TT^*\xi, \xi \rangle - |\langle T^*\xi, \xi \rangle|^2.$$

When T is self-adjoint this is the “mean-square deviation” of T in the state determined by ξ [35].

We now need to consider how small a neighborhood of $S(A)$ contains $S(B)$. Let $\nu \in S(B)$ be given. We choose $\mu = \nu \circ \check{\sigma}$, and observe that $\mu \in S(A)$. Let $(f, T) \in A \oplus B$ be such that $L(f, T) \leq 1$, so that $N_\sigma(f, T) \leq \gamma$. Then

$$\begin{aligned} |\mu(f, T) - \nu(f, T)| &= |\nu(\check{\sigma}_f) - \nu(T)| \leq \|\check{\sigma}_f - T\| \\ &= \left\| d \int f(x) \alpha_x(P) dx - d \int \alpha_x(P) T dx \right\| \\ &= d \left\| \int \alpha_x(P) (f(x)I - T) dx \right\| \leq d \int N_x(f, T) dx \\ &\leq d N_\sigma(f, T) \leq d\gamma. \end{aligned}$$

But the presence of d here causes us difficulties later, so we take another path, namely that used near the end of section 2 of [29]. We have

$$\begin{aligned} \|\check{\sigma}_f - T\| &\leq \|\check{\sigma}_f - \check{\sigma}(\sigma_T)\| + \|\check{\sigma}(\sigma_T) - T\| \\ &\leq \|f - \sigma_T\| + \|\check{\sigma}(\sigma_T) - T\| \leq \gamma^A + \|\check{\sigma}(\sigma_T) - T\|, \end{aligned}$$

where we have used that $\|f - \sigma_T\| \leq N_\sigma(f, T)$, as seen in the proof of Proposition 7.1. Notice that $T \mapsto \|\check{\sigma}(\sigma_T) - T\|$ is a seminorm on B which takes value 0 for $T = I$, and so drops to a seminorm on $\tilde{B} = B/\mathbb{C}I$, where L_B becomes a norm.

Notation 8.4. We set

$$\delta^B = \sup\{\|T - \check{\sigma}(\sigma_T)\| : L_B(T) \leq 1\}.$$

With this notation the above discussion gives:

Proposition 8.5. *Suppose that $\gamma \geq \gamma^A \vee \gamma^B$, so that L has L_A and L_B as quotients (where $L = L_A \vee L_B \vee \gamma^{-1}(N_\sigma \vee N_\sigma^*)$). Then $S(B)$ is in the $(\gamma^A + \delta^B)$ -neighborhood of $S(A)$.*

9. THE SET-UP FOR COMPACT LIE GROUPS

We now specialize to the case in which G is a compact connected semisimple Lie group. We use many of the techniques used in sections 6 and 7 of [29], and we usually use the notation established in sections 5 and 6 of [29]. We now review that notation. We let \mathfrak{g}_0 denote the Lie algebra of G , while \mathfrak{g} denotes the complexification of \mathfrak{g}_0 . We choose a maximal torus in G , with corresponding Cartan subalgebra of \mathfrak{g} , its set of roots, and a choice of positive roots. We let (U, \mathcal{H}) be an irreducible unitary representation of G , and we let U also denote the corresponding representation of \mathfrak{g} . We choose a highest weight vector,

ξ , for (U, \mathcal{H}) with $\|\xi\| = 1$. For any $n \in \mathbb{Z}_{\geq 1}$ we set $\xi^n = \xi^{\otimes n}$ in $\mathcal{H}^{\otimes n}$, and we let (U^n, \mathcal{H}^n) be the restriction of $U^{\otimes n}$ to the $U^{\otimes n}$ -invariant subspace, \mathcal{H}^n , of $\mathcal{H}^{\otimes n}$ which is generated by ξ^n . Then (U^n, \mathcal{H}^n) is an irreducible representation of G with highest weight vector ξ^n , and its highest weight is just n times the highest weight of (U, \mathcal{H}) . We denote the dimension of \mathcal{H}^n by d_n .

We let $B^n = \mathcal{L}(\mathcal{H}^n)$. The action of G on B^n by conjugation by U^n will be denoted simply by α . We assume that a continuous length function, ℓ , has been chosen for G , and we denote the corresponding C^* -metric on B^n by L_n^B . We let P^n denote the rank-one projection along ξ^n . Then the α -stability subgroup H for $P = P^1$ will also be the stability subgroup for each P^n . Let γ_n^A and γ_n^B be the constants defined in Propositions 8.1 and 8.2 but for P^n .

As done earlier, we let $A = C(G/H)$, and we let L_A be the seminorm on A for ℓ and the action of G . We can now state the main theorem of this paper.

Theorem 9.1. *Let notation be as above. Set $\gamma_n = \max\{\gamma_n^A, \gamma_n^B\}$ for each n , and let L_n be defined on $A \oplus B^n$ as in Proposition 7.1 but using γ_n . Then $L_n \in \mathcal{M}_C(L_A, L_B)$, and the sequence $\{L_n\}$ shows that the sequence $\{\text{prox}(A, B^n)\}$ converges to 0 as n goes to ∞ .*

The next three sections will be devoted to the proof of this theorem.

10. THE PROOF THAT $\gamma_n^A \rightarrow 0$

Consistent with the notation of Proposition 8.1, we have set

$$\gamma_n^A = d_n \int \rho_{G/H}(e, x) \|P^n \alpha_x(P^n)\| dx.$$

Proposition 10.1. *The sequence $\{\gamma_n^A\}$ converges to 0.*

Proof. For any two vectors η, ζ we let $\langle \eta, \zeta \rangle_0$ denote the rank-one operator that they determine. Then for any n we have

$$\begin{aligned} \|P^n \alpha_x(P^n)\| &= \|\langle \xi^n, \xi^n \rangle_0 \langle U_x^n \xi^n, U_x^n \xi^n \rangle_0\| \\ &= |\langle U_x^n \xi^n, \xi^n \rangle|^2 = |\langle U_x \xi, \xi \rangle|^n = \|P \alpha_x(P)\|^n. \end{aligned}$$

We use the analogous treatment given in lemma 3.3 and theorem 3.4 of [29], where we see that $d_n |\langle U_x \xi, \xi \rangle|^{2n} dx$ ($= d_n \|P^n \alpha_x(P^n)\|^2 dx$) is a probability measure on G/H , and that the sequence of these probability measures converges in the weak-* topology to the δ -measure on G/H supported at eH . Since $\rho_{G/H}(e, e) = 0$, it follows that the sequence

$d_n \int \rho_{G/H}(e, x) \|P^n \alpha_x(P^n)\|^2 dx$ converges to 0. Now

$$\begin{aligned} \gamma_{2n}^A &= d_{2n} \int \rho_{G/H}(e, x) \|P \alpha_x(P)\|^{2n} dx \\ &= (d_{2n}/d_n) d_n \int \rho_{G/H}(e, x) \|P^n \alpha_x(P^n)\|^2 dx, \end{aligned}$$

and so if we can show that (d_{2n}/d_n) is bounded, then we find that the sequence $\{\gamma_{2n}\}$ converges to 0. We use the Weyl dimension formula, as presented for example in theorem 4.14.6 of [36], to show that $\{d_{2n}/d_n\}$ is bounded. We let ω be the highest weight of U for our choice \mathcal{P} of positive roots. If one examines the dimension formula, it is evident that one only needs to use those positive roots α such that $\langle \omega, \alpha \rangle > 0$. We denote this set by \mathcal{P}_ω , and we denote its cardinality by p . It is clear that for any $n \in \mathbb{Z}_{>0}$ we have $\mathcal{P}_{n\omega} = \mathcal{P}_\omega$. The Weyl dimension formula then tells us that

$$d_n = \left(\prod \langle n\omega + \delta, \alpha \rangle \right) / \left(\prod \langle \delta, \alpha \rangle \right)$$

where both products are taken over \mathcal{P}_ω , and δ is half the sum of the positive roots. Thus

$$\begin{aligned} d_{2n}/d_n &= \left(\prod \langle 2n\omega + \delta, \alpha \rangle \right) / \left(\prod \langle n\omega + \delta, \alpha \rangle \right) \\ &= \prod (1 + \langle n\omega, \alpha \rangle / \langle n\omega + \delta, \alpha \rangle) \leq 2^p, \end{aligned}$$

so that the sequence d_{2n}/d_n is bounded as needed, and consequently the sequence $\{\gamma_{2n}^A\}$ converges to 0. In the same way, we find that $d_{n+1}/d_n \leq (1+n^{-1})^p$. Since $0 \leq \|P \alpha_x(P)\| \leq 1$, we have $\|P \alpha_x(P)\|^n \geq \|P \alpha_x(P)\|^{n+1}$. Thus the integrals defining γ_n^A are non-increasing. It follows that $\gamma_{2n+1}^A \leq (1+(2n)^{-1})^p \gamma_{2n}^A$. Since the sequence $\{\gamma_{2n}^A\}$ converges to 0, it follows that the sequence $\{\gamma_{2n+1}^A\}$ does also, so that the sequence $\{\gamma_n^A\}$ converges to 0. \square

11. PROPERTIES OF BEREZIN SYMBOLS

We now need results related to those given in sections 4 and 5 of [29], leading to the proof of theorem 6.1 of [29], and we will shortly also need theorem 6.1 of [29] itself. But Jeremy Sain has found a substantial simplification of the proof of theorem 6.1 of [29]. He gives his argument in section 4.4 of [33] in the more complicated context of quantum groups. We will use his arguments here in our present context. This will in particular provide Sain's proof of theorem 6.1 of [29].

As in [29], we denote the Berezin symbol map from B^n to $A = C(G/H)$ by σ^n . From theorem 3.1 of [29] we find that σ^n is injective because ξ^n is a highest weight vector. Consistent with the notation defined near the beginning of Section 8, we denote the adjoint of σ^n by $\check{\sigma}^n$. We let

$$(11.1) \quad \delta_n^A = \int_{G/H} \rho_{G/H}(e, x) d_n \operatorname{tr}(P^n \alpha_x(P^n)) dx.$$

In section 3 of [29] δ_n^A was denoted by γ_n , and theorem 3.4 of [29] shows both that the sequence $\{\delta_n^A\}$ converges to 0, and that

$$(11.2) \quad \|f - \sigma^n(\check{\sigma}^n(f))\|_\infty \leq \delta_n^A L_A(f)$$

for all $f \in A$ and all n . We remark that $\sigma^n \circ \check{\sigma}^n$ is often called the “Berezin transform” (for a given n).

As in section 4 of [29] we let \hat{G} denote the set of equivalence classes of irreducible unitary representations of G . For any finite subset \mathcal{S} of \hat{G} we let $A_{\mathcal{S}}$ and $B_{\mathcal{S}}^n$ denote the direct sum of the isotypic components of A and B^n for the representations in \mathcal{S} and for the actions of G on A and B^n (and similarly for actions on other Banach spaces). Since σ^n is equivariant, it carries $B_{\mathcal{S}}^n$ into $A_{\mathcal{S}}$. Since σ^n is injective, it follows that the dimension of $B_{\mathcal{S}}^n$ is no larger than that of $A_{\mathcal{S}}$, which is finite.

Since $\{\delta_n^A\}$ converges to 0, it follows from equation 11.2 that $\sigma^n \circ \check{\sigma}^n$ converges strongly to the identity operator on the space of functions f for which $L_A(f) < \infty$. But $A_{\mathcal{S}}$ is contained in this space and is finite-dimensional, and $\sigma^n \circ \check{\sigma}^n$ carries $A_{\mathcal{S}}$ into itself for each n . Consequently $\sigma^n \circ \check{\sigma}^n$ restricted to $A_{\mathcal{S}}$ converges in norm to the identity operator on $A_{\mathcal{S}}$. It follows that there is an integer, $N_{\mathcal{S}}$, such that $\sigma^n \circ \check{\sigma}^n$ on $A_{\mathcal{S}}$ is invertible and $\|(\sigma^n \circ \check{\sigma}^n)^{-1}\| < 2$ for every $n > N_{\mathcal{S}}$. In particular, σ^n from $B_{\mathcal{S}}^n$ to $A_{\mathcal{S}}$ will be surjective for $n > N_{\mathcal{S}}$. Since, as mentioned above, σ^n is always injective, and $\|\sigma^n\| = 1 = \|\check{\sigma}^n\|$ for all n , we can quickly see that:

Lemma 11.3. *(See corollary 4.17 of [33].) For $n > N_{\mathcal{S}}$ both σ^n and $\check{\sigma}^n$ going between $A_{\mathcal{S}}$ and $B_{\mathcal{S}}^n$ are invertible and their inverses have operator-norm no bigger than 2.*

Fix $n > N_{\mathcal{S}}$, and let $T \in B_{\mathcal{S}}^n$ be given. Set $f = (\check{\sigma}^n)^{-1}(T)$. Note that f is well-defined, and that $\|f\|_\infty \leq 2\|T\|$ by Lemma 11.3. Then

$$\|T - \check{\sigma}^n(\sigma^n(T))\| = \|\check{\sigma}^n(f) - \check{\sigma}^n(\sigma^n(\check{\sigma}^n(f)))\| \leq \|f - \sigma^n(\check{\sigma}^n(f))\| \leq \delta_n^A L_A(f),$$

where we have used inequality 11.2 for the last inequality above. Because $(\check{\sigma}^n)^{-1}$ is α - λ -equivariant and $\|(\check{\sigma}^n)^{-1}\| \leq 2$, we have $L_A(f) \leq 2L_n^B(T)$. We have thus obtained:

Lemma 11.4. (See proposition 4.19 of [33].) For any $n > N_S$ and any $T \in B_S^n$ we have

$$\|T - \check{\sigma}^n(\sigma_T^n)\| \leq 2\delta_n^A L_n^B(T).$$

Choose a faithful finite-dimensional unitary representation, π_0 , of G that contains the trivial representation, and let $\pi = \pi_0 \otimes \bar{\pi}_0$, where $\bar{\pi}_0$ is the contragradient representation for π_0 . Let χ be the character of π . Then χ is a non-negative real-valued function on G . Since π is faithful, we have the strict inequality $\chi(x) < \chi(e)$ for any $x \in G$ with $x \neq e$. Let χ^m denote the character of $\pi^{\otimes m}$, so that equally well it is the m^{th} pointwise power of χ . Set $\varphi_m = \chi^m / \|\chi^m\|_1$. Then the sequence $\{\varphi_m\}$ is a norm-1 approximate identity for the convolution algebra $L^1(G)$, as seen in the proof of theorem 8.2 of [28]. Furthermore, each φ_m is central in $L^1(G)$. Let β be an isometric strongly continuous action of G on a Banach space D , and let L^D be the corresponding seminorm for ℓ . Let β_{φ_n} denote the corresponding “integrated form” operator. As in the proof of lemma 8.3 of [28], for each $d \in D$ we have

$$\begin{aligned} \|d - \beta_{\varphi_m}(d)\| &= \left\| d \int \varphi_m(x) dx - \int \varphi_m(x) \beta_x(d) dx \right\| \\ &\leq \int \varphi_m \|d - \beta_x(d)\| dx \leq \left(\int \varphi_m(x) \ell(x) dx \right) L^D(d), \end{aligned}$$

and the sequence $\{\int \varphi_m(x) \ell(x) dx\}$ converges to 0.

We can now argue exactly as in the rest of the proof of theorem 6.1 of [29] to obtain:

Theorem 11.5. (Theorem 6.1 of [29]) For each $n \geq 1$ let δ_n^B be as defined in Notation 8.4 but for B^n , so that it is the smallest constant such that

$$\|T - \check{\sigma}^n(\sigma_T^n)\| \leq \delta_n^B L_n^B(T)$$

for all $T \in B^n$. Then the sequence $\{\delta_n^B\}$ converges to 0.

Proof of Theorem 11.5. Let $\varepsilon > 0$ be given. We can choose $\varphi = \varphi_m$ as just above such that for any ergodic action β of G on any unital C^* -algebra C we have $\|c - \beta_\varphi(c)\| \leq (\varepsilon/3)L(c)$ for all $c \in C$. Now φ is a positive function, and is a linear combination of the characters of a finite subset \mathcal{S} of \hat{G} , and so the integrated operator β_φ is a completely positive unital equivariant map of C onto its S -isotypic component.

Then for every n and every $T \in B^n$ we have $\alpha_\varphi(T) \in B_S^n$ and

$$\|T - \check{\sigma}^n(\sigma_T^n)\| \leq (\varepsilon/3)L_n^B(T) + \|\alpha_\varphi(T) - \check{\sigma}^n(\sigma_{\alpha_\varphi(T)}^n)\| + (\varepsilon/3)L_n^B(T).$$

From Lemma 11.4 there is an integer N_ε such that for any $n > N_\varepsilon$ and any $T' \in B_\mathcal{S}^n$ we have

$$\|T' - \check{\sigma}^n(\sigma^n(T'))\| \leq (\varepsilon/3)L_n^B(T').$$

Since $\alpha_\varphi(T) \in B_\mathcal{S}^n$, we can apply this to $T' = \alpha_\varphi(T)$. When we use the fact that $L_n^B(\alpha_\varphi(T)) \leq L_n^B(T)$, we see that for any $n > N_\varepsilon$ and any $T \in B^n$ we have

$$\|T - \check{\sigma}^n(\sigma_T^n)\| \leq \varepsilon L_n^B(T).$$

This immediately implies the statement about the sequence $\{\delta_n^B\}$. \square

12. THE PROOF THAT $\gamma_n^B \rightarrow 0$

Consistent with the notation of Proposition 8.2, we have set

$$\gamma_n^B = \sup\{\|P^n \operatorname{tr}(P^n T) - P^n T\| : T \in B^n \text{ and } L_n^B(T) \leq 1\}.$$

Proposition 12.1. *The sequence $\{\gamma_n^B\}$ converges to 0.*

Proof. Let $\varepsilon > 0$ be given. With the notation that we used just before Theorem 11.5, choose m_0 such that for $\varphi = \varphi_{m_0}$ we have $\int \varphi(x)\ell(x)dx \leq \varepsilon/4$. Then by the calculation done there we have

$$\|T - \alpha_\varphi(T)\| \leq (\varepsilon/4)L_n^B(T)$$

for all n and for all $T \in B^n$. Then for any n and any $T \in B^n$

$$\begin{aligned} & \| (P^n \operatorname{tr}(P^n T) - P^n T) - (P^n \operatorname{tr}(P^n \alpha_\varphi(T)) - P^n \alpha_\varphi(T)) \| \\ & \leq |\operatorname{tr}(P^n(T - \alpha_\varphi(T)))| + \|T - \alpha_\varphi(T)\| \\ & \leq 2\|T - \alpha_\varphi(T)\| \leq (\varepsilon/2)L_n^B(T), \end{aligned}$$

where for the next-to-last inequality we have used the fact that $P^n(T - \alpha_\varphi(T))$ is of rank 1.

Now as discussed in the proof of Theorem 11.5, φ is a linear combination of the characters of a finite subset \mathcal{S} of \hat{G} . Thus $\alpha_\varphi(T) \in B_\mathcal{S}^n$ and $L_n^B(\alpha_\varphi(T)) \leq L_n^B(T)$, and so we now see that it suffices to prove:

Main Lemma 12.2. *Let \mathcal{S} be given. For any $\varepsilon > 0$ there is an integer N_ε such that for any $n \geq N_\varepsilon$ and any $T \in B_\mathcal{S}^n$ we have*

$$\|P^n \operatorname{tr}(P^n T) - P^n T\| \leq (\varepsilon/2)L_n^B(T).$$

Proof. Let $f \in A$, and let n be given. Because A is commutative and $\check{\sigma}^n$ is positive, it follows from Kadison's generalized Schwarz inequality (e.g. 10.5.9 of [12]) that we have

$$\check{\sigma}_f^n(\check{\sigma}_f^n)^* \leq \check{\sigma}_{f\bar{f}}^n$$

for the usual order on positive operators. When we combine this with Proposition 8.3 we obtain

$$\begin{aligned} \|P^n(\operatorname{tr}(P^n \check{\sigma}_f^n)I - \check{\sigma}_f^n)\|^2 &= \operatorname{tr}(P^n \check{\sigma}_f^n (\check{\sigma}_f^n)^* P^n) - |\operatorname{tr}(P^n \check{\sigma}_f^n)|^2 \\ &\leq \operatorname{tr}(P^n \check{\sigma}_{ff}^n) - |\operatorname{tr}(P^n \check{\sigma}_f^n)|^2 = (\sigma^n(\check{\sigma}_{ff}^n))(e) - |\sigma^n(\check{\sigma}_f^n)(e)|^2, \end{aligned}$$

which by equation 11.2 above and theorem 3.4 of [29] converges to

$$(f\bar{f})(e) - |f(e)|^2 = 0$$

as n increases.

For each n define an operator, J^n , on B^n by

$$J^n(T) = P^n(\operatorname{tr}(P^n T)I - T).$$

The calculation above shows that the sequence $J^n(\check{\sigma}_f^n)$ converges to 0 for any $f \in A$ with $L^A(f) < \infty$. For \mathcal{S} as above it follows that the sequence of restrictions of $J^n \circ \check{\sigma}^n$ to $A_{\mathcal{S}}$ converges to 0 in operator norm. Let $N_{\mathcal{S}}$ be as in Lemma 11.3, so that $\|(\check{\sigma}^n)^{-1}\| \leq 2$ for $n > N_{\mathcal{S}}$. It follows that for $n > N_{\mathcal{S}}$ we have $\|J_n\| \leq 2\|J^n \circ \check{\sigma}^n\|$, so that the sequence of restrictions of J^n to $B_{\mathcal{S}}^n$ converges to 0 in norm. Thus for any $\varepsilon' > 0$ we can find an $n_{\varepsilon'}$ such that for $n > n_{\varepsilon'}$ and all $T \in B_{\mathcal{S}}^n$ we have

$$\|J^n(T)\| \leq \varepsilon' \|T\|.$$

Now $J^n(I) = 0$, and so it follows that

$$\|J^n(T)\| \leq \varepsilon' \|\tilde{T}\|^\sim,$$

where much as before $\|\cdot\|^\sim$ denotes the quotient norm on $\tilde{B}^n = B^n/\mathbb{C}I$. But by lemma 2.4 of [24] the radius of each of the algebras B^n is no larger than $r = \int \ell(x)dx$, in the sense that $\|\tilde{T}\|^\sim \leq rL_n^B(T)$ for all $T \in B^n$. We include a slightly simpler proof here. For $T \in B^n$ let $\eta(T) = \int \alpha_x(T) dx$, so that $\eta(T) \in \mathbb{C}I$ since U^n is irreducible. Then

$$\|\tilde{T}\|^\sim \leq \|T - \eta(T)\| = \left\| \int (T - \alpha_x(T))dx \right\| \leq L_n^B(T) \int \ell(x)dx.$$

It follows that

$$J^n(T) \leq r\varepsilon' L_n^B(T).$$

Consequently, if we choose $\varepsilon' = \varepsilon/(2r)$, and set $N_\varepsilon = n_{\varepsilon'} \vee N_{\mathcal{S}}$, we find that for $n \geq N_\varepsilon$ we have

$$\|P^n \operatorname{tr}(P^n T) - P^n T\| \leq (\varepsilon/2)L_n^B(T)$$

for all $T \in B_{\mathcal{S}}^n$, as needed. □

□

13. THE PROOF OF THE MAIN THEOREM

We now use the results of the previous sections to prove Theorem 9.1. For any n set $\gamma_n = \max(\gamma_n^A, \gamma_n^B)$, and define L_n on $A \oplus B^n$ by

$$L_n(f, T) = L_A(f) \vee L_n^B(T) \vee \gamma_n^{-1}(N_{\sigma^n}(f, T) \vee N_{\sigma^n}(\bar{f}, T^*)).$$

Then for each n we have $\gamma_n \geq \gamma_n^A$ so that the quotient of L_n on A is L_A by Proposition 8.1, and we have $\gamma_n \geq \gamma_n^B$ so that the quotient of L_n on B^n is L_n^B by Proposition 8.2. Thus L_n is in $\mathcal{M}_C(L_A, L_n^B)$ as defined in Notation 5.5.

Then according to Proposition 7.1 (with notation as in Proposition 8.1 and in the sentence before Proposition 10.1), $S(A)$ is in the γ_n -neighborhood of $S(B^n)$ for ρ_{L_n} . Furthermore, according to Proposition 8.5 (with notation as in Theorem 11.5) $S(B^n)$ is in the $(\gamma_n^A + \delta_n^B)$ -neighborhood of $S(A)$. It follows that

$$\text{dist}_H^{\rho_{L_n}}(S(A), S(B^n)) \leq \max\{\gamma_n^A + \delta_n^B, \gamma_n\} \leq \max\{\gamma_n^A + \delta_n^B, \gamma_n^B\},$$

and so

$$\text{prox}(A, B^n) \leq \max\{\gamma_n^A + \delta_n^B, \gamma_n^B\}.$$

But γ_n^A , δ_n^B and γ_n^B all converge to 0 as n goes to ∞ , according to Proposition 10.1, Theorem 11.5 (theorem 6.1 of [29]), and Proposition 12.1 respectively. Consequently $\text{prox}(A, B^n)$ converges to 0 as n goes to ∞ , as desired.

14. MATRICIAL SEMINORMS

In this section we will briefly describe the relations between the previous sections of this paper and several variants of quantum Gromov–Hausdorff distance.

The first variant is the matricial quantum Gromov–Hausdorff distance introduced by Kerr [13]. It has the advantage that if two C^* -algebras with Lip-norms are at distance 0 for this distance then the C^* -algebras are isomorphic. We will not repeat here Kerr’s definitions and results for general operator systems; rather we will only indicate, somewhat sketchily, what Kerr’s variant says in the context of the present paper. For any unital C^* -algebra A and each $q \in \mathbb{Z}_{>0}$ the $*$ -algebra $M_q(A)$ of $q \times q$ matrices with entries in A has a unique C^* -norm. The collection of these C^* -norms forms a “matricial norm” for A . Given unital C^* -algebras A and B , a linear map $\varphi : A \rightarrow B$ determines for each q a linear map, φ^q , from $M_q(A)$ to $M_q(B)$, by entry-wise application. One says that φ is “completely positive” if each φ^q is positive as a map between C^* -algebras. For each q let $UCP_q(A)$ denote the collection of unital completely positive maps from A into $M_q(\mathbb{C})$.

The $UCP_q(A)$'s are called the “matricial state-spaces” of A . All these considerations apply equally well to unital C^* -normed algebras, where “positive” is with respect to the completions.

Let a Lip^* -norm, L , on A be specified. Then Kerr defines a metric, ρ_L^q , on $UCP_q(A)$ by

$$\rho_L^q(\varphi, \psi) = \sup\{\|\varphi(a) - \psi(a)\| : L(a) \leq 1\},$$

and he shows that the topology on $UCP_q(A)$ determined by ρ_L^q agrees with the point-norm topology (and so is compact). Now let (A, L_A) and (B, L_B) be unital C^* -algebras with Lip^* -norms. Essentially as in definition 4.2 of [28] let $\mathcal{M}(L_A, L_B)$ denote the set of Lip^* -norms on $A \oplus B$ whose quotients on the self-adjoint part agree with L_A and L_B . Note that $UCP_n(A)$ and $UCP_n(B)$ can be viewed as subsets of $UCP_n(A \oplus B)$ in an evident way. Then for each q Kerr defines the q -distance, dist_s^q , between A and B by

$$\text{dist}_s^q(A, B) = \inf\{\text{dist}_H^{\rho_L^q}(UCP_q(A), UCP_q(B)) : L \in \mathcal{M}(A, B)\},$$

and he defines the complete distance, dist_s , by

$$\text{dist}_s(A, B) = \sup_q \{\text{dist}_s^q(A, B)\}.$$

Finally (for our purposes), he shows that for our setting of coadjoint orbits with $A = C(G/H)$ and $B^n = \mathcal{L}(\mathcal{H}_n)$ with their Lip^* -norms from a length function ℓ , one has

$$\lim_{n \rightarrow \infty} \text{dist}_s(A, B^n) = 0.$$

We can quickly adapt Kerr's arguments to our Leibniz setting. For C^* -algebras A and B equipped with C^* -metrics, we define $\mathcal{M}_C(L_A, L_B)$ exactly as in Notation 5.5. Any L in $\mathcal{M}_C(L_A, L_B)$ is, in particular, a Lip^* -norm, and so defines for each q the metric ρ_L^q on $UCP_q(A \oplus B)$. We can then define, for each q ,

$$\text{prox}^q(A, B) = \inf\{\text{dist}_H^{\rho_L^q}(UCP_q(A), UCP_q(B)) : L \in \mathcal{M}_C(A \oplus B)\}.$$

Then we can define “complete proximity” by

$$\text{prox}_s(A, B) = \sup_q \{\text{prox}^q(A, B)\}.$$

Of course, we have

$$\text{dist}_s(A, B) \leq \text{prox}_s(A, B).$$

Theorem 14.1. *For $A = C(G/H)$ and $B^n = \mathcal{L}(\mathcal{H}_n)$ with their C^* -metrics L_A and L_B^n as defined earlier in terms of a length function on G , we have*

$$\lim_{n \rightarrow \infty} \text{prox}_s(A, B^n) = 0.$$

Proof. We follow the outline of Kerr’s example 3.13 of [13], but for a given n we set, as earlier,

$$L_n = L_A \vee L_n^B \vee N_n \vee N_n^*$$

with $N_n = \gamma_n^{-1} N_{\sigma^n}$ and with γ_n chosen exactly as in the proof of Theorem 9.1 that is completed in Section 13. Thus $L_n \in \mathcal{M}_C(L_A, L_n^B)$. The key observation, for Kerr and for us, is that σ^n and $\check{\sigma}^n$ are (unital) completely positive maps, so that if $\varphi \in UCP_q(A)$ then $\varphi \circ \sigma^n$ is in $UCP_q(B^n)$, and similarly for $\check{\sigma}^n$. Given $\varphi \in UCP_q(A)$, set $\psi = \varphi \circ \sigma^n$. Then exactly as in the proof of Proposition 7.1 we see that if $L_n(f, T) \leq 1$ then

$$\|\varphi(f) - \psi(T)\| \leq \|f - \sigma_T\| \leq \gamma_n,$$

so that $UCP_q(A)$ is in the γ_n neighborhood of $UCP_q(B^n)$. On the other hand, for any $\psi \in UCP_q(B^n)$ set $\varphi = \psi \circ \check{\sigma}$. Then in the somewhat more complicated way given in Section 13 we find that $UCP_q(B^n)$ is in as small a neighborhood of $UCP_q(A)$ as desired if n is sufficiently large. \square

We remark that in section 5 of [13] Kerr considers a weak form of the Leibniz property which he calls “ f -Leibniz” (for which he comments that the corresponding distance may not satisfy the triangle inequality).

In [17] Hanfeng Li introduced a quite flexible variant of quantum Gromov–Hausdorff distance that in a suitable way uses the Hausdorff distance between the unit L -balls of two quantum metric spaces. Li called this “order-unit quantum Gromov–Hausdorff distance”. In [14] Kerr and Li developed a matricial version of Li’s variant, which they called “operator Gromov–Hausdorff distance”. They then show (theorem 3.7) that this version coincides with Kerr’s matricial quantum Gromov–Hausdorff distance. It would be interesting to have a version of our complete proximity above that is defined in terms of the unit L -balls, since it might well have certain technical advantages similar to those possessed by Li’s order-unit Gromov–Hausdorff distance.

For the specific case of C^* -algebras, Li introduced [18] yet another variant of quantum Gromov–Hausdorff distance that explicitly uses the algebra multiplication. He calls this “ C^* -algebraic quantum Gromov–Hausdorff distance”. It would be interesting to know how this version relates to Leibniz seminorms and proximity. We should mention that

in several places the later papers of Kerr and of Li discussed in this section again consider the f -Leibniz property that Kerr introduced in [13].

Hanfeng Li has pointed out to me that much the same arguments as given in the last part of Section 6 showing that prox is dominated by his dist_{nu} , also show that our “complete proximity” prox_s is dominated by dist_{nu} ; and since, as mentioned in Section 6, the examples that have been studied so far for convergence for quantum Gromov-Hausdorff distance all involve nuclear C^* -algebras, and convergence for them holds for dist_{nu} , this gives for them a proof of convergence for prox_s .

The papers discussed above all begin just with a Lip-norm. In a different direction Wei Wu has defined and studied matricial Lipschitz seminorms [38, 39, 40]. Again, we will not repeat here his general definitions and results; rather we will only indicate somewhat sketchily how they can be adapted to the context of the present paper, I thank Wei Wu for answering several questions that I had about his papers.

Let G be a compact group equipped with a length function ℓ , and let α be an action of G on a unital C^* -algebra A . Then G has an evident entry-wise action on $M_q(A)$ for each $q \in \mathbb{Z}_{>0}$, and we can then use ℓ to define a seminorm, L^q , on each $M_q(A)$ as in Example 2.5. This family of seminorms satisfies Ruan-type axioms [10], in particular, $L(T_{ij}) \leq L^q(T)$ for $T = \{T_{ij}\} \in M_q(A)$. Wu presents this family as one example of what he calls a “matrix Lipschitz seminorm” on A . It is a very natural example, and it indicates how natural it is to consider matrix Lipschitz seminorms quite generally. However Wu does not make use of the fact that each of the seminorms L^q above is Leibniz (in fact, strongly Leibniz), and he uses the bridge from [29], which is not Leibniz.

For $A = C(G/H)$ and B^n as earlier we denote the seminorms by L_A^q and $L_B^{n,q}$. As Wu notes, the Berezin symbol map σ^n gives, by entry-wise application, a completely positive map from $M_q(B^n)$ to $M_q(A)$ for each $q \in \mathbb{Z}_{>0}$. We denote these maps still by σ^n . Much as in Section 7 we can then define a seminorm on $M_q(A \oplus B^n)$ by

$$\|M_f \circ \sigma^n - \sigma^n \circ \Lambda_T\|.$$

But the analogue of the alternative description in terms of seminorms N_x given in Proposition 7.2 is now more complicated, and so I have found it best just to work directly with the analogs of the N_x ’s. Specifically, we write $\text{diag}(\alpha_x(P^n))$ for the matrix in $M_q(B^n)$ each of whose diagonal entries is $\alpha_x(P^n)$, with all other entries being 0. For each

$x \in G$ (or G/H) we set

$$N_x^{n,q}(f, T) = \|\text{diag}(\alpha_x(P))(f(x) \otimes I_n - T)\|$$

for any $(f, T) \in M_q(A \oplus B^n)$. It is easily seen that $N_x^{n,q}$ is strongly Leibniz. We then set

$$N_\sigma^{n,q}(f, T) = \sup\{N_x^{n,q}(f, T) : x \in G\}.$$

Then we set

$$N_{n,q}(f, T) = \gamma^{-1} N_\sigma^{n,q}(f, T),$$

where γ remains to be chosen for each n . Finally we set

$$L_{n,q}(f, T) = L_A^q(f) \vee L_B^{n,q}(T) \vee N_{n,q}(f, T) \vee N_{n,q}^*(f, T).$$

It is easily verified that the family $\{L_{n,q}\}$ is a “matrix Lipschitz seminorm” as defined in definition 3.1 of [40]. We would like to choose γ in such a way that the quotients of $L_{n,q}$ on $M_q(A)$ and $M_q(B^n)$ are L_A^q and $L_B^{n,q}$.

We consider the quotient on $M_q(A)$ first. We note, as does Wu, that $\check{\sigma}^n$ gives, by entry-wise application, a unital completely positive map from $M_q(A)$ to $M_q(B^n)$. Given $f \in M_q(A)$, we set $T = \check{\sigma}_f^n$. Then, much as in Section 8,

$$N_x^{n,q}(f, T) = \left\| \left\{ \alpha_x(P^n)(f_{ij}(x)I_n - d \int f_{ij}(y)\alpha_y(P^n)dy) \right\} \right\|,$$

where $\{\cdot\}$ denotes a matrix. As in Section 8, the translation-invariance of L_A^q and the arbitrariness of f permit us to consider just the case in which $x = e$. Then, with manipulations as in Section 8, we see that

$$\begin{aligned} N_e^{n,q}(f, T) &\leq d \int \|\{f_{ij}(e) - f_{ij}(y)\}\| \|\text{diag}(P^n \alpha_y(P^n))\| dy \\ &\leq L_A^q(f) \int \rho(e, y) d\|P^n \alpha_y(P^n)\| dy \\ &= L_A^q(f) \gamma_n^A, \end{aligned}$$

where γ_n^A is defined at the beginning of Section 9. Thus if $\gamma \geq \gamma_n^A$ then the quotient of $L_{n,q}$ on $M_q(A)$ will be L_A^q , which is exactly the same condition as for the case of $q = 1$ treated in Section 8.

We now consider the quotient on $M_q(B^n)$. Given $T \in M_q(B^n)$, we set $f = \sigma_T^n$. Then

$$N_x^{n,q}(f, T) = \|\{\alpha_x(P)(\text{tr}(\alpha_x(P)T_{ij})I_n - T_{ij})\}\|.$$

I don't see a good way to estimate this except by the entry-wise estimate

$$\leq q \sup_{i,j} \|\alpha_x(P)(\text{tr}(\alpha_x(P)T_{ij})I_n - T_{ij})\|$$

$$\leq q\gamma_n^B \sup_{i,j} L_n^B(T_{ij}) \leq q\gamma_n^B L_B^{n,q}(T),$$

where γ_n^B is defined at the beginning of Section 12, and where we have used the α -invariance of L_n^B , and the fact that for any $R \in M_q(B^n)$ we have $\|R\| \leq q \sup_{i,j} \{\|R_{ij}\|\}$. (To see this latter, express R as the sum of the q matrices whose only non-zero entries are the entries R_{ij} of R for which $i - j$ is constant modulo q .) Thus if $\gamma \geq q\gamma_n^B$ then the quotient of $L_{n,q}$ on $M_q(B^n)$ will be $L_B^{n,q}$. The factor of q in this estimate has the quite undesirable effect that we seem not to be able to say that for a sufficiently large γ it is true that for all q simultaneously the quotient of $L_{n,q}$ on $M_q(B^n)$ is $L_B^{n,q}$. Thus the family $\{L_{n,q}\}$ can not be used to estimate the “quantized Gromov–Hausdorff distance” defined by Wu in definition 4.5 of [40]. But for fixed q we will still have that $q\gamma_n^B$ converges to 0 as $n \rightarrow \infty$, and this may still be useful, for instance in dealing with vector bundles along the lines discussed in [31].

According to Wu’s definition of “quantized Gromov–Hausdorff distance” we must now show that $UCP_q(A)$ and $UCP_q(B^n)$ are within suitable neighborhoods of each other in $UCP_q(A \oplus B)$ (once we have chosen $\gamma \geq \gamma_n^A \vee q\gamma_n^B$). Given $f \in M_q(A)$ and $\varphi \in UCP_q(A)$ (which Wu denotes by $CS_q(A)$), let $\langle\langle \varphi, f \rangle\rangle$ denote the element of $M_{q^2}(\mathbb{C})$ whose entries are the $\varphi_{ij}(f_{kl})$ ’s. (See 1.1.27 of [10].) Equivalently, view f as in $M_q \otimes A$, and let $\tilde{\varphi} = I_q \otimes \varphi$ so that $\tilde{\varphi} : M_q \otimes A \rightarrow M_q \otimes M_q$. Then $\langle\langle \varphi, f \rangle\rangle = \tilde{\varphi}(f)$. We can thus use L_A^q to define a metric, $D_{L_A^q}$, on $UCP_q(A)$, defined by

$$D_{L_A^q}(\varphi_1, \varphi_2) = \sup\{\|\langle\langle \varphi_1, f \rangle\rangle - \langle\langle \varphi_2, f \rangle\rangle\| : f \in M_q(A), L_A^q(f) \leq 1\}.$$

(See proposition 3.1 of [39].) Wu shows that the topology on $UCP_q(A)$ from the metric $D_{L_A^q}$ coincides with the point-norm topology. In the same way $L_B^{n,q}$ defines a metric on $UCP_q(B^n)$, and $L_{n,q}$ defines a metric on $UCP_q(A \oplus B^n)$. Furthermore, when we view $UCP_q(A)$ and $UCP_q(B^n)$ as subsets of $UCP_q(A \oplus B^n)$, the restriction of $D_{L_{n,q}}$ to them will agree with $D_{L_A^q}$ and $D_{L_B^{n,q}}$ if the quotients of $L_{n,q}$ on $M_q(A)$ and $M_q(B^n)$ agree with L_A^q and $L_B^{n,q}$. (See proposition 3.6 of [40].)

We now show that $UCP_q(A)$ is in a suitably small neighborhood of $UCP_q(B^n)$ for $D_{L_n^q}$.

Lemma 14.2. *For any $(f, T) \in M_q(A \oplus B^n)$ we have*

$$\|f - \sigma_T^n\| \leq qN_\sigma^{n,q}(f, T).$$

Proof.

$$\|f - \sigma_T^n\| = \sup_x \|\{f_{ij}(x) - \text{tr}(\alpha_x(P)T_{ij})\}\|$$

$$\begin{aligned}
&\leq q \sup_{x,i,j} |\operatorname{tr}(\alpha_x(P)(f_{ij}(x)I_n - T_{ij}))| \\
&\leq q \sup_{x,i,j} \|\alpha_x(P)(f_{ij}(x)I_n - T_{ij})\| \\
&\leq q \sup_x \|\{\alpha_x(P)(f_{ij}(x)I_n - T_{ij})\}\| = qN_\sigma^{n,q}(f, T).
\end{aligned}$$

□

We can now proceed much as in the first half of Wu's proof of theorem 8.6 of [40]. Let q be fixed, and now set $\gamma_n = \gamma_n^A \vee q\gamma_n^B$ in the definition of $L_{n,q}$, so that $L_{n,q}$ has the right quotients. Let $\varphi \in UCP_q(A)$ be given. Set $\psi = \varphi \circ \sigma^n$, so that $\psi \in UCP_q(B^n)$. Suppose that $(f, T) \in M_q(A \oplus B^n)$ and that $L_n^q(f, T) \leq 1$, so that $N_\sigma^{n,q}(f, T) \leq \gamma_n$. Then by Lemma 14.2

$$\begin{aligned}
\|\langle\langle\varphi, f\rangle\rangle - \langle\langle\psi, T\rangle\rangle\| &= \|\langle\langle\varphi, f - \sigma_T^n\rangle\rangle\| \\
&\leq \|f - \sigma_T^n\| \leq qN_\sigma^{n,q}(f, T) \leq q\gamma_n.
\end{aligned}$$

Thus $UCP_q(A)$ is in the $q\gamma_n$ -neighborhood of $UCP_q(B^n)$. Since $\gamma_n^A \vee q\gamma_n^B$ converges to 0 as $n \rightarrow \infty$ we can make $q\gamma_n$ as small as desired by choosing n large enough.

We now show that $UCP_q(B^n)$ is in a suitably small neighborhood of $UCP_q(A)$. We can proceed as in the second half of Wu's proof of his theorem 8.6 of [40]. Let $\psi \in UCP_q(B^n)$ be given. Set $\varphi = \psi \circ \check{\sigma}^n$, so that $\varphi \in UCP_q(A)$. For $L(f, T) \leq 1$ as above we have, much as in the proof of Proposition 8.5,

$$\begin{aligned}
\|\langle\langle\varphi, f\rangle\rangle - \langle\langle\psi, T\rangle\rangle\| &= \|\langle\langle\psi, \check{\sigma}_f^n - T\rangle\rangle\| \\
&\leq \|\check{\sigma}_f^n - T\| \leq \|\check{\sigma}_f^n - \check{\sigma}^n(\sigma_T^n)\| + \|\check{\sigma}^n(\sigma_T^n) - T\| \\
&\leq \|f - \sigma_T^n\| + \|\check{\sigma}^n(\sigma_T^n) - T\| \\
&\leq q\gamma_n + \|\check{\sigma}^n(\sigma_T^n) - T\|.
\end{aligned}$$

We can deal with the second of these terms much as we do in Section 13, just as Wu does. One then sees that for a given $\varepsilon > 0$ one can (for a fixed q) choose N large enough that $UCP_q(A)$ and $UCP_q(B^n)$ are in each other's ε -neighborhood for $n \geq N$.

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